

Read the following instructions before the exam.

### Format & How to Submit Answers

**There are 16 problems (4 introductory questions and 12 exam questions) of varying numbers of points.** The problems are of varying difficulty, so pace yourself accordingly and avoid spending too much time on any one question until you have gotten all of the other points you can. Don't get bogged down in calculations; if you are having trouble with one problem, there may be easier points available later in the exam!

All answers will be submitted to the Gradescope "Final Exam" Assignment (<https://www.gradescope.com/courses/137582/assignments/593619>). Questions are multiple choice, short answer, or free response. **There are 210 points possible on the exam, but your final score will be taken out of 195 points.** This means that a score of 160/210, normally 76%, will be bumped up to 160/195, or 82%. You cannot score more than 100% on this exam.

Partial credit may be given for certain incorrect answer choices for some multiple choice problems. There is no penalty for incorrect answers. Partial credit will be available for most free response questions.

Post any content or clarifying questions privately on Piazza. There will be no exam clarifications; if we find a bug on the exam, that sub-question will be omitted from grading.

**Free Response Questions:** You will scan and submit 9 free response subparts to the "Final Exam" assignment as you would a homework assignment. You should plan to have a stable internet connection during the three-hour exam period, as it will be necessary to save and submit your work. You are not required to print anything out to take this exam; you are required to have something to write on, something to write with, and a way to scan written work to a PDF or image file on a desktop or laptop computer.

If you'd like, you can do all of the free response questions on the same PDF and upload it (with all free response answers) to each of the free response subparts - just make sure each subpart is clearly labeled. If you're having trouble uploading your answers to Gradescope, you can also email them to [eecs16a@berkeley.edu](mailto:eecs16a@berkeley.edu), before the exam period (see below) is over.

We recommend you test your system for uploading a handwritten image before the Final Exam begins. Please refer to this YouTube video for instructions on uploading your PDF or image files. [https://youtu.be/j\\_kha2UkeMY?t=144](https://youtu.be/j_kha2UkeMY?t=144)

## Timing & Penalties

You have 180 minutes for the exam, with a 5 minute grace period. After the 5 minute grace period ends, exam scores will be penalized exponentially as follows: an exam that is submitted  $N$  minutes after the end of the grace period will lose  $2^N$  points. **The exam will become available at your personalized link at 7:10 pm PT; the grace period will expire at 10:15 pm PT.** If your submission is timestamped at 10:16 pm PT, you will lose 2 points; if it is timestamped at 10:18 pm PT, you will lose 8 points.

**We will count the latest time at which you submit any question as your exam timestamp.** Do NOT edit or resubmit your answers after the deadline. We recommend having all of your answers input and submitted by 10:10 pm; it is your responsibility to submit the exam on time.

If you cannot access your exam at your link by 7:15 pm, please email [eeecs16a@berkeley.edu](mailto:eeecs16a@berkeley.edu). If you are having technical difficulties submitting your exam, you can email your answers (either typed or scanned) to [eeecs16a@berkeley.edu](mailto:eeecs16a@berkeley.edu).

## Academic Honesty

This is an open-note, open-book, open-internet, and **closed-neighbor** exam. You may use any calculator or calculation software that you wish, including Wolfram-Alpha and Mathematica. **No collaboration is allowed, and do not attempt to cheat in any way. Cheating will not be tolerated.**

**We have zero tolerance against violation of the Berkeley Honor Code. Given supporting evidence of cheating, we reserve the right to automatically fail all students involved and report the instance to the student conduct committee.** We reserve the right to **audit students with oral exams** after the midterm to ensure academic honesty. Feel free to report suspicious activity through this form. (<https://forms.gle/ZzXLksZEmx9bn1mj7>).

**Our advice to you:** if you can't solve a particular problem, move on to another, or state and solve a simpler one that captures at least some of its essence. You will perhaps find yourself on a path to the solution.

**Good luck!**

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EECS 16A    Designing Information Devices and Systems I  
Summer 2020    Midterm 2

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**1. Pledge of Academic Integrity (2 points)**

By my honor, I affirm that:

- (1) this document, which I will produce for the evaluation of my performance, will reflect my original, bona fide work;
- (2) as a member of the UC Berkeley community, I have acted and will act with honesty, integrity, and respect for others;
- (3) I have not violated—nor aided or abetted anyone else to violate—nor will I—the instructions for this exam given by the course staff, including, but not limited to, those on the cover page of this document; and
- (4) I have not committed, nor will I commit, any act that violates—nor aided or abetted anyone else to violate—the UC Berkeley Code of Student Conduct.

Write your name and the current date as an acknowledgement of the above. (See Gradescope)

**2. Administrivia (1 point)**

I know that I will lose  $2^n$  points for every  $n$  minutes I submit after the exam submission grace period is over.

For example, if the exam becomes available at my personalized link at 7:10 p.m. PT; the grace period will expire at 10:15 p.m. PT. If my submission is timestamped at 10:16 p.m. PT, I will lose 2 points; if it is timestamped at 10:18 p.m. PT, I will lose 8 points.

- Yes

**3. What have you enjoyed most about EECS16A? (2 points)**

**4. Tell us about something you're looking forward to! (2 points)**

### 5. Least Squares Fitting (9 points)

In an upward career move, you join the starship USS Enterprise as a data scientist. One morning the Chief Science Officer, Mr. Spock, hands you some data for the position ( $y$ ) of a newly discovered particle at different times ( $t$ ). The data has four points and **contains some noise**:

$$(t = 0, y = 5), \quad (t = 1, y = 7.1), \quad (t = 2, y = 12.5), \quad (t = 3, y = 19)$$

Your research shows that the path of the particle is represented by the function:

$$y = w_1 e^t + w_2 e^{-2t} + \sin(t) + w_3 t \quad (1)$$

You decide to fit the collected data to the function in Equation (1) using the Least Squares method.

- (a) (3 points) You need to find the coefficients  $w_1$ ,  $w_2$ , and  $w_3$  that *minimize the squared error* between the fitted curve and the collected data points. So you set up a system of linear equations,  $\mathbf{A}\hat{\alpha} \approx \vec{b}$  in order

to find the approximate value of  $\hat{\alpha} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ . Select the correct setup.

(A) 
$$\begin{bmatrix} e^0 & e^0 & \sin(0) + 0 \\ e^1 & e^{-2} & \sin(1) + 1 \\ e^2 & e^{-4} & \sin(2) + 2 \\ e^3 & e^{-6} & \sin(3) + 3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \approx \begin{bmatrix} 5 \\ 7.1 \\ 12.5 \\ 19 \end{bmatrix}$$

(B) None of these options are correct.

(C) 
$$\begin{bmatrix} e^0 & e^0 & \sin(0) & 0 \\ e^1 & e^{-2} & \sin(1) & 1 \\ e^2 & e^{-4} & \sin(2) & 2 \\ e^3 & e^{-6} & \sin(3) & 3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \approx \begin{bmatrix} 5 \\ 7.1 \\ 12.5 \\ 19 \end{bmatrix}$$

(D) 
$$\begin{bmatrix} e^0 & e^0 & 0 \\ e^1 & e^{-2} & 1 \\ e^2 & e^{-4} & 2 \\ e^3 & e^{-6} & 3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \approx \begin{bmatrix} 5 \\ 7.1 \\ 12.5 \\ 19 \end{bmatrix}$$

(E) 
$$\begin{bmatrix} e^0 & e^0 & 0 \\ e^1 & e^{-2} & 1 \\ e^2 & e^{-4} & 2 \\ e^3 & e^{-6} & 3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \approx \begin{bmatrix} 5 - \sin(0) \\ 7.1 - \sin(1) \\ 12.5 - \sin(2) \\ 19 - \sin(3) \end{bmatrix}$$

**Solution:** For  $t = 0$ , we have:

$$\begin{aligned} 5 &= w_1 e^0 + w_2 e^{-2 \cdot 0} + \sin(0) + w_3 \cdot 0 \\ \implies 5 - \sin(0) &= w_1 e^0 + w_2 e^0 + w_3 \cdot 0 \end{aligned}$$

Similarly for  $t = 1, 2$ , and  $3$ , we have:

$$\begin{aligned} 7.1 - \sin(1) &= w_1 e^1 + w_2 e^{-2} + w_3 \\ 12.5 - \sin(2) &= w_1 e^2 + w_2 e^{-4} + 2w_3 \\ 19 - \sin(3) &= w_1 e^3 + w_2 e^{-6} + 3w_3 \end{aligned}$$

Hence we can write the following system of linear equations:

$$\begin{bmatrix} e^0 & e^0 & 0 \\ e^1 & e^{-2} & 1 \\ e^2 & e^{-4} & 2 \\ e^3 & e^{-6} & 3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \approx \begin{bmatrix} 5 - \sin(0) \\ 7.1 - \sin(1) \\ 12.5 - \sin(2) \\ 19 - \sin(3) \end{bmatrix}$$

(b) (3 points) Mr. Spock thinks one of the data points is wrong and asks you to redo the fit with only three data points. What will happen to the norm of the error,  $\|\vec{e}\| = \|\vec{b} - \mathbf{A}\hat{\vec{\alpha}}\|$ ?

- (A) There will be an infinite number of possible fits.
- (B)  $\|\vec{e}\|$  will be zero.
- (C)  $\|\vec{e}\|$  will decrease, but will have a non-zero value.
- (D) The data cannot be fit.
- (E)  $\|\vec{e}\|$  will increase.
- (F)  $\|\vec{e}\|$  might increase or decrease. There is not enough information to tell.

**Solution:** The linear system now has three unknowns ( $w_1$ ,  $w_2$ , and  $w_3$ ) and three linearly-independent constraints (the three data points), so there will be an exact fit to the data: the norm of error  $\|\vec{e}\| = \|\vec{b} - \mathbf{A}\hat{\vec{\alpha}}\|$  will be 0. This is probably too good to be true!

- (c) (3 points) Your colleague tries to repeat your fitting process with the same four data points in part (a), but they misread the equation relating  $t$  and  $y$ , i.e. they use the following function (which is **different than part (a)**):

$$y = w_1 e^t + w_2 e^t + \sin(t) + w_3 t \quad (2)$$

Your colleague tries to find  $w_1$ ,  $w_2$  and  $w_3$  by setting up a system of equations  $\mathbf{A}\hat{\alpha} \approx \vec{b}$  and utilizing the equation:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \hat{\alpha} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \vec{b}. \quad (3)$$

What is the most likely outcome of Equation (3)?

- (A) They won't find  $\hat{\alpha}$ , because the new equation does not correspond to the data.
- (B) They will find  $\hat{\alpha}$ , and  $\|\vec{e}\|$  will be zero.
- (C) They won't find  $\hat{\alpha}$ , because  $\mathbf{A}$  has linearly dependent columns.
- (D) They will find  $\hat{\alpha}$ , but  $\|\vec{e}\|$  will be larger than part (a).
- (E) There is not enough information to decide the outcome.

**Solution:** The new system of linear equations can be written as:  $\begin{bmatrix} e^0 & e^0 & 0 \\ e^1 & e^1 & 1 \\ e^2 & e^2 & 2 \\ e^3 & e^3 & 3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \approx \begin{bmatrix} 5 - \sin(0) \\ 7.1 - \sin(1) \\ 12.5 - \sin(2) \\ 19 - \sin(3) \end{bmatrix}$

. This makes the first and second columns of  $\mathbf{A}$  exactly same. Since  $\mathbf{A}$  has linearly dependent columns,  $\mathbf{A}^T \mathbf{A}$  will not be invertible, i.e. the equation for  $\hat{\alpha}$  will not work. The correct answer is (C).

**6. Project Projections (9 points)**

(a) (3 points) Find  $\text{proj}_{\mathbb{S}}(\vec{v})$ , the projection of vector  $\vec{v}$  onto the subspace  $\mathbb{S}$ .

$$\vec{v} = \begin{bmatrix} 3 \\ 9 \\ 0 \end{bmatrix} \quad \mathbb{S} = \text{span} \left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right\}$$

**Solution:** First we observe that we are given an orthogonal basis for subspace  $\mathbb{S}$ . Therefore, to calculate the projection, we can project  $\vec{v}$  onto the two basis vectors individually.

Let the basis vectors be called  $\vec{s}_1$  and  $\vec{s}_2$ .

$$\begin{aligned} \text{proj}_{\mathbb{S}}(\vec{v}) &= \text{proj}_{\vec{s}_1}(\vec{v}) + \text{proj}_{\vec{s}_2}(\vec{v}) = \frac{\langle \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \rangle}{\langle \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \rangle} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \frac{\langle \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \rangle}{\langle \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \rangle} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \\ &= -b \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \frac{3a}{10} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{9a}{10} \\ b \\ -\frac{3a}{10} \end{bmatrix} \\ &= \begin{bmatrix} 2.7 \\ 9.0 \\ -0.9 \end{bmatrix} \end{aligned}$$

(b) (3 points) Let  $\hat{\vec{v}}$  be an approximation of  $\vec{v}$ . We compute different approximations using the expressions below:

$$\vec{v} = \begin{bmatrix} 6 \\ 10 \\ -7 \end{bmatrix}, \quad \mathbb{S}_1 = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ -8 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \right\}, \quad \mathbb{S}_2 = \text{span} \left\{ \begin{bmatrix} 3 \\ x \\ y \end{bmatrix}, \begin{bmatrix} z \\ -9 \\ 5 \end{bmatrix} \right\}, \quad \mathbf{A} = \begin{bmatrix} 3 & z \\ x & -9 \\ y & 5 \end{bmatrix}$$

$$\hat{\vec{v}}_{\mathbb{S}_1} := \text{proj}_{\mathbb{S}_1}(\vec{v}) \quad \hat{\vec{v}}_{\mathbb{S}_2} := \text{proj}_{\mathbb{S}_2}(\vec{v}) \quad \hat{\vec{v}}_{\mathbf{A}} := \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \vec{v}$$

The error vectors for each of these approximations are

$$\vec{e}_{\mathbb{S}_1} := \vec{v} - \hat{\vec{v}}_{\mathbb{S}_1} \quad \vec{e}_{\mathbb{S}_2} := \vec{v} - \hat{\vec{v}}_{\mathbb{S}_2} \quad \vec{e}_{\mathbf{A}} := \vec{v} - \hat{\vec{v}}_{\mathbf{A}}$$

Rank the *norms* of the error vectors, from smallest to largest. Assume that  $x, y, z$  are values such that  $\vec{v} \notin \mathbb{S}_2$ .

**Solution:** The vector  $\vec{v}$  must be in  $\mathbb{S}_1$  since this subspace spans all of  $\mathbb{R}^3$ . Therefore the error from this projection will be 0.

Next, we are told that  $\vec{v}$  is not in  $\mathbb{S}_2$ . This means that if we project  $\vec{v}$  onto  $\mathbb{S}_2$ , we will get some non-zero error. We know that an alternative way to calculate projections when we are given a basis for the subspace is through least squares. We see that  $\hat{\vec{v}}_{\mathbb{A}}$  is the least squares calculation for the projection onto  $\mathbb{S}_2$ . Therefore, it will yield the same error as  $\hat{\vec{v}}_{\mathbb{S}_2}$ .

$$\|\vec{e}_{\mathbb{S}_1}\| < \|\vec{e}_{\mathbb{S}_2}\| = \|\vec{e}_{\mathbb{A}}\|$$

- (c) (3 points) Let  $\alpha$  and  $\beta$  be scalars, and let  $\vec{p}$  be a vector such that  $\text{proj}_{\vec{q}}(\vec{p}) = \vec{w}$ . Find an expression for  $\text{proj}_{\beta\vec{q}}(\alpha\vec{p})$  in terms of  $\alpha$ ,  $\beta$ , and  $\vec{w}$ .

**Solution:** We start by substituting  $\alpha\vec{p}$  and  $\beta\vec{q}$  into the projection equation.

$$\frac{\langle \alpha\vec{p}, \beta\vec{q} \rangle}{\langle \beta\vec{q}, \beta\vec{q} \rangle} \beta\vec{q} = \frac{\alpha\beta\langle \vec{p}, \vec{q} \rangle}{\beta\langle \vec{q}, \vec{q} \rangle} \vec{q} = \alpha\vec{w}$$

Therefore,  $\text{proj}_{\beta\vec{q}}(\alpha\vec{p}) = \alpha\vec{w}$ .



## 7. Essay Plagiarism Catcher (12 points)

Your lab TA, Rohan, has decided to quit engineering and teach an English course instead, and he assigns an essay for the first homework. He decides to put his EECS 16A knowledge to use by building a plagiarism detection system using linear algebra.

- (a) (3 points) Rohan would like to find out if any of the students copied part of an example essay that he wrote. He uses an online tool “Essay2Vec” to convert the example essay to a vector:

$$\vec{v}_{\text{example}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

He also uses “Essay2Vec” to convert the students’ essays into vectors,  $\vec{v}_{\text{student}}$ .

Rohan determines that if  $\vec{v}_{\text{student}}$  makes an angle of less than  $30^\circ$  with  $\vec{v}_{\text{example}}$ , then he should check the essay for plagiarism by hand. By this criteria, which of the following submitted essays should Rohan check? Select all that apply.

*Hint:*  $\cos 0^\circ = 1$ ,  $\cos 90^\circ = 0$ ,  $\cos 30^\circ = \frac{\sqrt{3}}{2} = 0.866$ .

(1)  $\vec{v}_{\text{student}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(2)  $\vec{v}_{\text{student}} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

- (3) None of these  $\vec{v}_{\text{student}}$  essays need to be checked.

(4)  $\vec{v}_{\text{student}} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

**Solution:** We can use the inner product formula to find the angle between any vectors. We rearrange the formula  $\langle \vec{a}, \vec{b} \rangle = \|\vec{a}\| \|\vec{b}\| \cos(\theta)$  to find that  $\cos \theta = \frac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{a}\| \|\vec{b}\|} \implies \theta = \arccos\left(\frac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{a}\| \|\vec{b}\|}\right)$ . For

$\vec{v}_{\text{student}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ , we have

$$\cos \theta_1 = \frac{\left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\rangle}{\left\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\| \left\| \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\|} = \frac{4}{\sqrt{18}} = 0.94 > \cos 30^\circ$$

$$\implies \theta_1 = \arccos(0.94) \approx 19.5^\circ < 30^\circ$$

Similarly, for  $\vec{v}_{\text{student}} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ , we can see that  $\vec{v}_{\text{student}}$  and  $\vec{v}_{\text{essay}}$  are orthogonal, i.e. their inner product

is zero:

$$\cos \theta_2 = \frac{\left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\rangle}{\left\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\| \left\| \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\|} = \frac{0}{\sqrt{18}} = 0$$

$$\implies \theta_2 = \arccos(0) = 90^\circ > 30^\circ$$

For  $\vec{v}_{\text{student}} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ , we see that  $\vec{v}_{\text{student}}$  and  $\vec{v}_{\text{essay}}$  are scalar multiple of each other, i.e. they are on the same line:

$$\cos \theta_4 = \frac{\left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\rangle}{\left\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\| \left\| \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\|} = \frac{6}{\sqrt{36}} = 1$$

$$\implies \theta_4 = \arccos(1) = 0^\circ < 30^\circ$$

Therefore, options (1) and (4) make angles of less than  $30^\circ$  with  $\vec{v}_{\text{example}}$ .

(b) (3 points) Rohan next calculates the difference,  $\vec{d}$ , between the student essays and his example essay:

$$\vec{d} = \vec{v}_{\text{student}} - \vec{v}_{\text{example}}$$

Rohan expects  $\vec{d}$  to have a small magnitude for a plagiarized essay. Let

$$\vec{v}_{\text{example}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v}_{\text{student}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

If  $\theta$  is the angle between  $\vec{v}_{\text{student}}$  and  $\vec{v}_{\text{example}}$ , find the relationship between  $\|\vec{d}\|^2$  and  $\theta$ .

(A)  $\|\vec{d}\|^2 = \sqrt{3} + \sqrt{6} - 2\sqrt{18} \cos \theta$

(B)  $\|\vec{d}\|^2 = 9 - 2\sqrt{18} \cos \theta$

(C)  $\|\vec{d}\|^2 = \sqrt{3} + \sqrt{6} - \sqrt{18} \cos \theta$

(D)  $\|\vec{d}\|^2 = 9 + \sqrt{18} \cos \theta$

(E)  $\|\vec{d}\|^2 = 9 - \sqrt{18} \cos \theta$

(F)  $\|\vec{d}\|^2 = 9 + 2\sqrt{18} \cos \theta$

**Solution:**

$$\begin{aligned} \|\vec{d}\|^2 &= \langle \vec{d}, \vec{d} \rangle \\ &= \langle \vec{v}_{\text{student}} - \vec{v}_{\text{example}}, \vec{v}_{\text{student}} - \vec{v}_{\text{example}} \rangle \\ &= (\vec{v}_{\text{student}} - \vec{v}_{\text{example}})^T (\vec{v}_{\text{student}} - \vec{v}_{\text{example}}) \\ &= (\vec{v}_{\text{student}}^T - \vec{v}_{\text{example}}^T) (\vec{v}_{\text{student}} - \vec{v}_{\text{example}}) \\ &= \vec{v}_{\text{student}}^T \vec{v}_{\text{student}} + \vec{v}_{\text{example}}^T \vec{v}_{\text{example}} - \vec{v}_{\text{student}}^T \vec{v}_{\text{example}} - \vec{v}_{\text{example}}^T \vec{v}_{\text{student}} \\ &= \|\vec{v}_{\text{student}}\|^2 + \|\vec{v}_{\text{example}}\|^2 - 2\langle \vec{v}_{\text{student}}, \vec{v}_{\text{example}} \rangle \end{aligned}$$

Here  $\|\vec{v}_{\text{example}}\| = \sqrt{3}$ ,  $\|\vec{v}_{\text{student}}\| = \sqrt{6}$  and  $\langle \vec{v}_{\text{student}}, \vec{v}_{\text{example}} \rangle = \|\vec{v}_{\text{student}}\| \|\vec{v}_{\text{example}}\| \cos \theta = \sqrt{3}\sqrt{6} \cos \theta$ .  
 Plugging in these values, we get:

$$\begin{aligned} \|\vec{d}\|^2 &= 3 + 6 - 2\sqrt{3}\sqrt{6} \cos \theta \\ &= 9 - 2\sqrt{18} \cos \theta \end{aligned}$$

- (c) (3 points) Rohan considers the set  $\mathbb{S}$  of **all possible essay vectors**  $\vec{v}_{\text{student}}$  with  $\|\vec{d}\| < \sqrt{10}$ . In other words,

$$\mathbb{S} = \{ \vec{v}_{\text{student}} \}, \quad \text{such that} \quad \|\vec{v}_{\text{student}} - \vec{v}_{\text{example}}\| < \sqrt{10}$$

Is  $\mathbb{S}$  a subspace of  $\mathbb{R}^3$ ? Assume  $\vec{v}_{\text{example}}$  is the same as the previous two parts.

- (A) No
- (B) There is not enough information to determine.
- (C) Yes

**Solution:**  $\mathbb{S}$  will be a subspace of  $\mathbb{R}^3$ , if it is closed under vector addition, closed under scalar multiplication and contains the zero vector. For  $\vec{v}_{\text{student}} = \vec{0}$ ,  $\|\vec{v}_{\text{student}} - \vec{v}_{\text{example}}\| = \|\vec{0} - \vec{v}_{\text{example}}\| = \sqrt{3} < \sqrt{10}$  So zero vector is contained in  $\mathbb{S}$ .

However,  $\mathbb{S}$  is not closed under scalar multiplication. For example,  $\vec{v}_{\text{student}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$  is in  $\mathbb{S}$ , since

$$\|\vec{v}_{\text{student}} - \vec{v}_{\text{example}}\| = \left\| \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\| = 1 < \sqrt{10}. \text{ However, a scalar multiple } 3\vec{v}_{\text{student}} = 3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$$

is not in  $\mathbb{S}$  since  $\|\vec{v}_{\text{student}} - \vec{v}_{\text{example}}\| = \left\| \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} \right\| = \sqrt{33} \notin \sqrt{10}$ .

$\mathbb{S}$  is also not closed under vector addition. For example  $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$  are both in  $\mathbb{S}$ , however,  $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} +$

$$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \text{ is not in } \mathbb{S}, \text{ since } \left\| \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \right\| = \sqrt{27} \notin \sqrt{10}.$$

- (d) (3 points) Rohan knows a student included a paragraph from the textbook in their essay, but he doesn't know which page of the essay it's on.

He assigns the letters A-Z to integers 1-26 and creates a vector  $\vec{s} \in \mathbb{R}^{10,000}$  representing the student essay and a vector  $\vec{p} \in \mathbb{R}^{500}$  representing the paragraph from the textbook. He uses Python to calculate  $\text{corr}_{\vec{s}}(\vec{p})$  and searches for a peak in the cross-correlation.

Will the index of the highest peak definitely correspond to the location of the textbook paragraph in the essay? Why or why not?

- (A) Yes, because the highest peak is where the angle between  $\vec{s}$  and  $\vec{p}$  is maximized.
- (B) No, because the method used above will generally give higher peaks where the student essay contains many letters towards the end of the alphabet.

- (C) Yes, because Rohan knows that  $\vec{p}$  is somewhere in  $\vec{s}$ , so he is guaranteed to find one peak corresponding to the correct location.
- (D) No, because the greatest similarity occurs at the greatest magnitude of the dot product, which could be a negative value.
- (E) Yes, because the highest peak indicates the maximum dot product, which means the vectors are the most similar at that index.
- (F) No, because the method used above will generally give higher peaks where the student essay contains many letters towards the beginning of the alphabet.

**Solution:** The cross-correlation peak corresponds to the best match of the signals when the norm of the signals of the same length are the same. Any part of the essay of length 500, which contains many letters towards the end of the alphabet will have a higher norm. So its inner product with  $\vec{p}$  will yield a higher value even though it might not match  $\vec{p}$ .

**8. Determinants (15 points)**

For any square matrix  $\mathbf{A}$ , we can compute  $\det(\mathbf{A} - \lambda\mathbf{I})$ . Parts (a) - (e) below each give a different expression for  $\det(\mathbf{A} - \lambda\mathbf{I})$ , and your job is to determine which of seven possible statements must be true, given this information. Select all that apply:

(a) (2 points)  $\det(\mathbf{A} - \lambda\mathbf{I}) = (3 - \lambda)^2$

- (1)  $\mathbf{A}$  is a  $2 \times 2$  matrix
- (2)  $\mathbf{A}$  is a  $3 \times 3$  matrix
- (3)  $\mathbf{A}$  has a nontrivial nullspace
- (4)  $\mathbf{A}$  is invertible
- (5) The dimension of at least one of the eigenspaces cannot be determined from the given information
- (6)  $\mathbf{A}$  has complex eigenvalues (with nonzero imaginary component)
- (7)  $\mathbf{A}$  is the zero matrix

**Solution:** Because the characteristic equation is a polynomial of degree 2,  $\mathbf{A}$  must be a  $2 \times 2$  matrix. This characteristic equation has roots  $\lambda = 3$  with multiplicity 2. Since there are no nonzero eigenvalues,  $\mathbf{A}$  has a trivial nullspace and therefore invertible. Additionally, the multiplicity of two means there is the potential for a 2-dimensional eigenspace. **In summary, (1) (4) (5) are true.**

(b) (2 points)  $\det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda(5 - \lambda)^2$

- (1)  $\mathbf{A}$  is a  $2 \times 2$  matrix
- (2)  $\mathbf{A}$  is a  $3 \times 3$  matrix
- (3)  $\mathbf{A}$  has a nontrivial nullspace
- (4)  $\mathbf{A}$  is invertible
- (5) The dimension of at least one of the eigenspaces cannot be determined from the given information
- (6)  $\mathbf{A}$  has complex eigenvalues (with nonzero imaginary component)
- (7)  $\mathbf{A}$  is the zero matrix

**Solution:** Because the characteristic equation is a polynomial of degree 3, the  $\mathbf{A}$  is a  $3 \times 3$  matrix. This characteristic equation has roots  $\lambda = 5$  with multiplicity 2 and  $\lambda = 0$ . Since there is an eigenvalue of zero,  $\mathbf{A}$  has a nontrivial nullspace. Additionally, the multiplicity of two means there is the potential for a 2-dimensional eigenspace. **In summary, (2) (3) (5) are true.**

(c) (2 points)  $\det(\mathbf{A} - \lambda\mathbf{I}) = (14 - \lambda)(3 - \lambda)(9 - \lambda)$

- (1)  $\mathbf{A}$  is a  $2 \times 2$  matrix
- (2)  $\mathbf{A}$  is a  $3 \times 3$  matrix
- (3)  $\mathbf{A}$  has a nontrivial nullspace
- (4)  $\mathbf{A}$  is invertible
- (5) The dimension of at least one of the eigenspaces cannot be determined from the given information
- (6)  $\mathbf{A}$  has complex eigenvalues (with nonzero imaginary component)
- (7)  $\mathbf{A}$  is the zero matrix

**Solution:** Because the characteristic equation is a polynomial of degree 3, the  $\mathbf{A}$  is a  $3 \times 3$  matrix. This characteristic equation has roots  $\lambda = 14, 3, 9$ . Since there are no nonzero eigenvalues,  $\mathbf{A}$  has a trivial nullspace and therefore invertible. **In summary, (2) (4) are true.**

(d) (2 points)  $\det(\mathbf{A} - \lambda \mathbf{I}) = -\lambda(\lambda^2 + \lambda + 9)$

- (1)  $\mathbf{A}$  is a  $2 \times 2$  matrix
- (2)  $\mathbf{A}$  is a  $3 \times 3$  matrix
- (3)  $\mathbf{A}$  has a nontrivial nullspace
- (4)  $\mathbf{A}$  is invertible
- (5) The dimension of at least one of the eigenspaces cannot be determined from the given information
- (6)  $\mathbf{A}$  has complex eigenvalues (with nonzero imaginary component)
- (7)  $\mathbf{A}$  is the zero matrix

**Solution:** Because the characteristic equation is a polynomial of degree 3, the  $\mathbf{A}$  is a  $3 \times 3$  matrix. This characteristic equation has roots  $\lambda = 0$  and  $\lambda = \frac{-1 \pm i\sqrt{35}}{2}$ , obtained using the quadratic formula. Since there is an eigenvalue of zero,  $\mathbf{A}$  has a nontrivial nullspace. Additionally, the roots with imaginary number  $i$  are complex. **In summary, (2) (3) (6) are true.**

(e) (2 points)  $\det(\mathbf{A} - \lambda \mathbf{I}) = \lambda^2$

- (1)  $\mathbf{A}$  is a  $2 \times 2$  matrix
- (2)  $\mathbf{A}$  is a  $3 \times 3$  matrix
- (3)  $\mathbf{A}$  has a nontrivial nullspace
- (4)  $\mathbf{A}$  is invertible
- (5) The dimension of at least one of the eigenspaces cannot be determined from the given information
- (6)  $\mathbf{A}$  has complex eigenvalues (with nonzero imaginary component)
- (7)  $\mathbf{A}$  is the zero matrix

**Solution:** Because the characteristic equation is a polynomial of degree 2, the  $\mathbf{A}$  is a  $2 \times 2$  matrix. This characteristic equation has roots  $\lambda = 0$  with multiplicity 2. Since there is an eigenvalue of zero,  $\mathbf{A}$  has a nontrivial nullspace. Additionally, the multiplicity of two means there is the potential for a 2-dimensional eigenspace. However, this characteristic equation does not mean that  $\mathbf{A}$  is the zero matrix. For example the matrix,  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  has the characteristic equation  $\lambda^2$  but is not the zero matrix. **In summary, (1) (3) (5) are true.**

(f) (5 points) Let  $\mathbf{A}$  be a  $2 \times 2$  matrix. Prove that if the columns of  $\mathbf{A}$  are linearly dependent, then  $\det(\mathbf{A}) = 0$ . Start with the following:

$$\mathbf{A} = \begin{bmatrix} | & | \\ \vec{x} & \vec{y} \\ | & | \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

**Solution:** To prove that  $\det(\mathbf{A}) = 0$  if the columns of  $\mathbf{A}$  are linearly dependent, we start with the definition of linear dependence. Since there are only two columns, we know that one must be a multiple of the other.

$$\vec{x} = \alpha \vec{y} \text{ or } x_1 = \alpha y_1, x_2 = \alpha y_2$$

Then we must recall our definition of the determinant of a  $2 \times 2$  matrix:

$$\det(\mathbf{A}) = x_1 y_2 - x_2 y_1$$

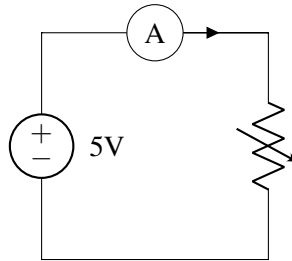
From there, we can plug in our first statement.

$$\det(\mathbf{A}) = \alpha y_1 y_2 - \alpha y_1 y_2 = 0$$

### 9. Straining to Understand (18 points)

Your friend in mechanical engineering wants your help fixing their device which measures mechanical forces and material strength. (The device is called a *strain gauge sensor*.)

- (a) (4 points) Your friend has made a test setup to measure how the resistance of their device changes as a function of force. The device datasheet specifies two parameters,  $R_0$ , the resistance without force, measured in  $\Omega$ 's, and  $\Delta R$ , the change in resistance per pound of force, measured in  $\frac{\Omega}{lb}$ 's. The test setup appears as follows:



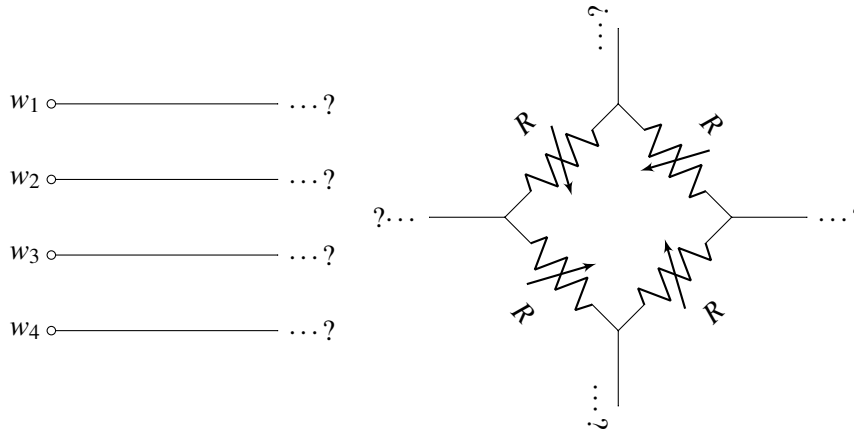
When no force is applied, your ammeter reads  $10\text{ mA}$ . When  $5\text{ lbs}$  are applied, your ammeter reads  $8\text{ mA}$ . What are the parameters,  $R_0$  and  $\Delta R$  of the strain gauge?

**Solution:** The resistance when no force is applied is given by Ohm's law as:  $R_0 = \frac{5V}{10mA} = 500\Omega$ .

The total resistance with force applied is equal to:  $R_f = \frac{5V}{8mA} = 625\Omega$

The change in resistance per pound of force is given by:  $\Delta R = \frac{625\Omega - 500\Omega}{5lb} = 25\frac{\Omega}{lb}$ .

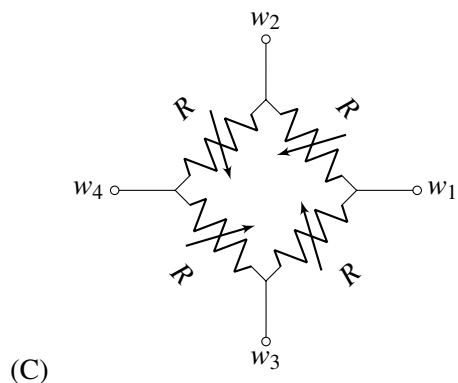
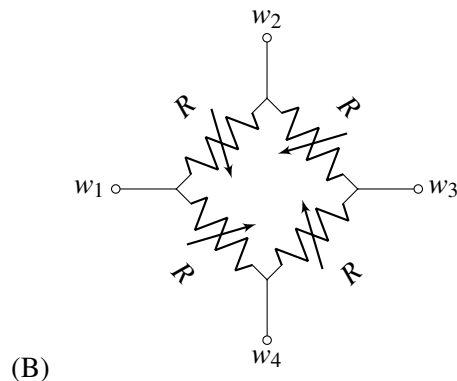
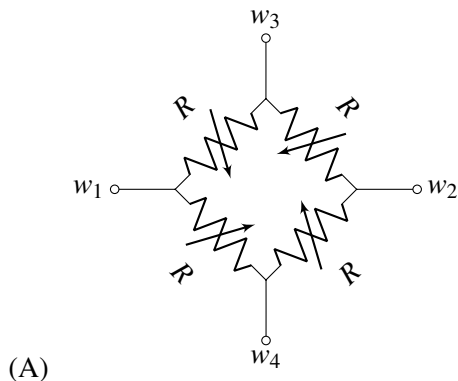
(b) (3 points) Your friend is glad to see how precise the resistors are. They assemble a full test setup with four of these devices. While testing this new circuit, they manage to destroy the connector going to the device. Thankfully, the wires labeled  $w_1$ ,  $w_2$ ,  $w_3$ , and  $w_4$  are still intact, but they no longer know which wire goes to which node in the circuit, depicted in the figure below. All resistors have a value of  $R = 680\Omega$ .



In an attempt to figure out which terminal might be which, using your multimeter you measure the equivalent resistance seen between pairs of wires and get the following reading:

$R_{w_1-w_2}$	$R_{w_1-w_3}$	$R_{w_1-w_4}$	$R_{w_2-w_3}$	$R_{w_2-w_4}$	$R_{w_3-w_4}$
680.0 $\Omega$	510.0 $\Omega$	510.0 $\Omega$	510.0 $\Omega$	510.0 $\Omega$	680.0 $\Omega$

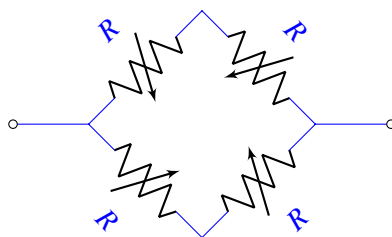
Select the circuit that represents the connections:



(D) None of these options are correct.



**Solution:** The only way to get  $680\Omega$  is to measure two terminals on opposite sides of the network that do not connect to the a common resistor seen as below:



The equivalent resistance between the terminals is:

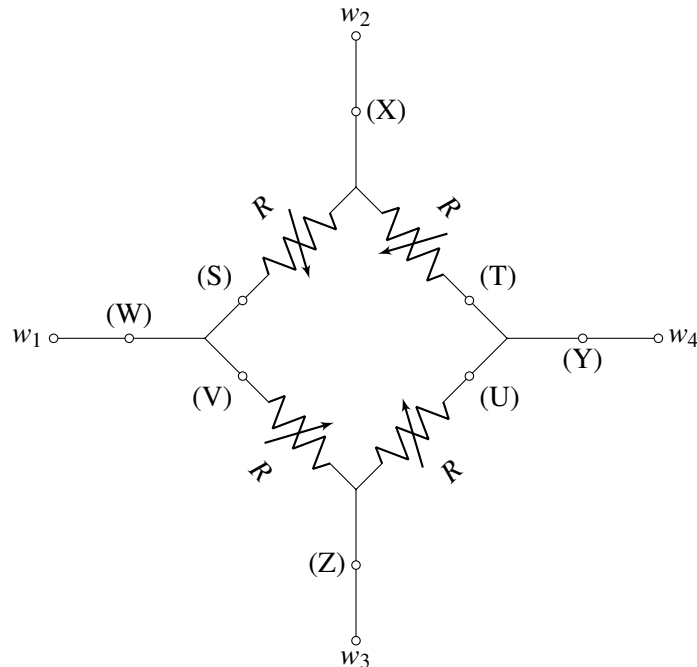
$$R_{eq} = (R + R) \parallel (R + R) = 2R \parallel 2R = R = 680\Omega$$

Any pair of terminals with a common resistor will have an equivalent resistance of:

$$R_{eq} = R \parallel (R + R + R) = R \parallel 3R = \frac{3}{4}R = 510\Omega$$

Since  $R_{w_1-w_2} = 680\Omega$  and  $R_{w_3-w_4} = 680\Omega$ , terminals 1 and 2 are on the opposite sides of the network. Also terminals 3 and 4 are also on opposite sides.

(c) (3 points) Congrats! You've identified the terminals and informed your friend how to figure out the configuration if the issue happens again. A week later, they come back to you with the following circuit, which may or may not be the correct answer to part (b). After a round of destructive testing, they found a single open in the circuit somewhere below (at one of these points: (S), (T), (U), (V), (W), (X), (Y), (Z)), but they're not quite sure where. All resistors have a value of  $R = 680\Omega$ , as before.



They want your help determining the location of the open circuit. You make the following equivalent resistance measurements. Your multimeter reads O.L. when it detects an open circuit.

$R_{w_1-w_2}(\Omega)$	$R_{w_1-w_3}(\Omega)$	$R_{w_1-w_4}(\Omega)$	$R_{w_2-w_3}(\Omega)$	$R_{w_2-w_4}(\Omega)$	$R_{w_3-w_4}(\Omega)$
O.L.	O.L.	O.L.	680.0	510.0	510.0

Where is the open circuit, i.e. where is the break located? Select the node with the break.

**Solution:** It appears that there are open loops whenever terminal  $w_1$  is paired with any other terminal. This narrows down our choices to (W), (S), and (V). It cannot be either of (S) and (V) as in that case one of the pairs  $w_1 - w_2$ ,  $w_1 - w_3$  would show some resistance across it instead of O.L. This leaves us with only option (W) remaining.

- (d) (8 points) Now assume that the resistors used depend on the applied pressure through the following affine function:

$$R = R_0 + \alpha P,$$

where  $R$  is the total resistance,  $P$  is the applied pressure you are trying to detect,  $R_0$  is the resistance when no pressure is applied, and  $\alpha$  is a real valued coefficient.

Your friend has **2** resistors with positive  $\alpha$ 's and **2** resistors with negative  $\alpha$ 's, denoted  $R_A$  and  $R_B$  respectively. Both resistor pairs have the same  $R_0$  but **equal and opposite**  $\alpha$ 's. Using the configuration below, how should they connect the four resistors if they want to get an output voltage,  $V_{out}$ , that will be a linear (not affine) function of pressure?

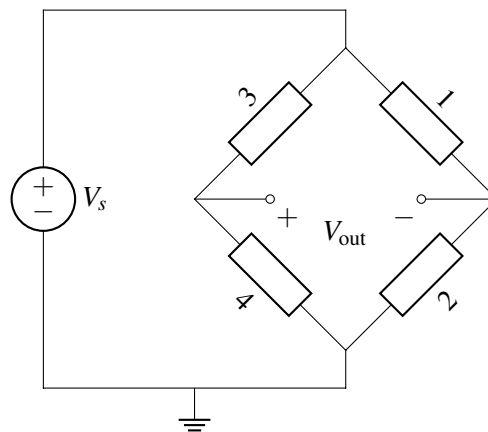
**Redraw** the circuit, replacing boxes 1 - 4 with the resistors  $R_A$  and  $R_B$ , in the proper locations in order to achieve this goal. **Calculate the absolute value** of the output voltage of the circuit you have come up with to show that it is indeed a linear function of pressure.

**Note:** there is more than one correct solution to this problem.

$$R_A = R_0 + \alpha P$$

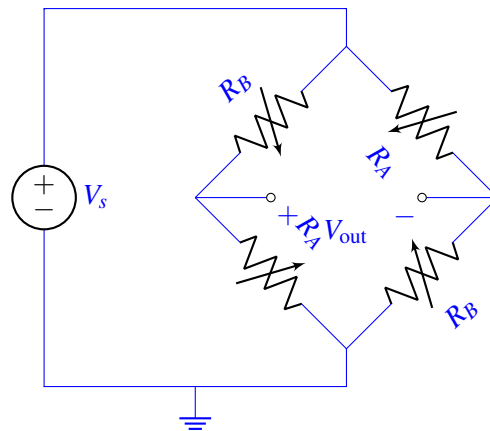
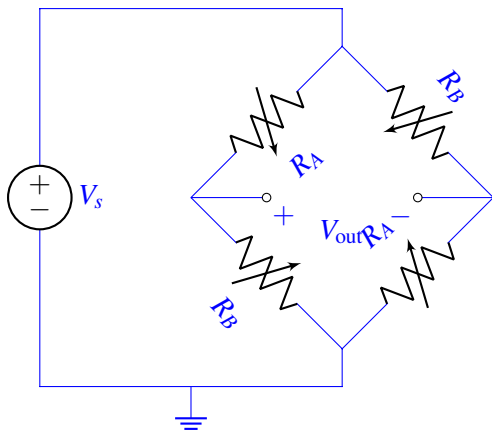


$$R_B = R_0 - \alpha P$$



**Solution:**

The two possible solutions are:



Let's analyze the circuit shown on the left (the circuit on the right will have  $V_{\text{out, right}} = -V_{\text{out, left}}$ , but the absolute value of the output voltage will be the same for both circuits).

We can see that this circuit is essentially composed of two voltage dividers, the difference of which is taken as our output. Hence, we can write that:

$$V_{\text{out}} = \frac{R_B}{R_A + R_B} V_s - \frac{R_A}{R_A + R_B} V_s = \frac{R_B - R_A}{R_A + R_B} V_s = \frac{(R_0 - \alpha P) - (R_0 + \alpha P)}{(R_0 - \alpha P) + (R_0 + \alpha P)} V_s = \frac{-2\alpha P}{2R_0} V_s$$
$$|V_{\text{out}}| = \frac{\alpha P}{R_0} V_s$$

**10. One Ambitious Crossover (10 points)** Pumped up by a summer full of 16A knowledge, you open up a freelancer business called Odd Jobs, along with your trusty sidekicks Specs and Tiny. Your very first client wants you to fix their antique AM radio!

- (a) (3 points) You need a circuit producing an output voltage  $V_{out}$ , which is a quadratic function of the input voltage  $V_{in}$ . Specs, the circuitry wiz in your team, brings in a non-linear device, as shown in Figure 1. The current through this device  $I_D$  depends on the voltage  $V_D$  across its terminals in the following manner:

$$I_D = c_1 V_D^2 + c_2 V_D + c_3$$

Here  $c_1$ ,  $c_2$  and  $c_3$  are scalars.

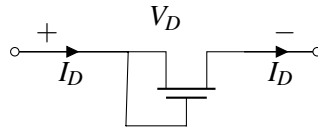


Figure 1: Non-linear device showing  $I_D$  and  $V_D$

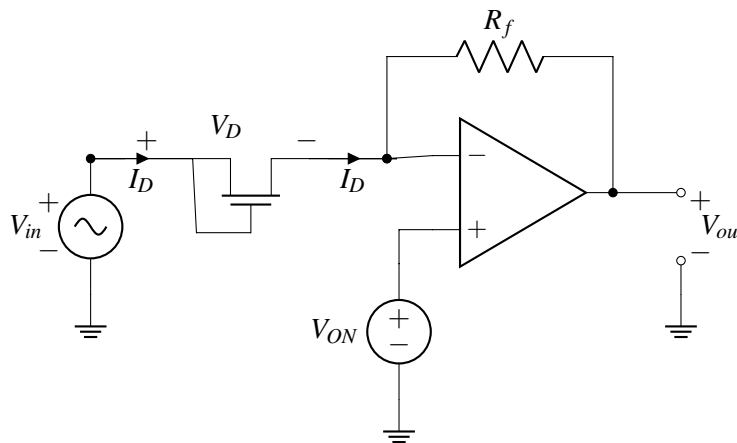


Figure 2: Circuit with an op-amp and the non-linear device, showing  $V_{in}$  and  $V_{out}$

Specs constructs the circuit shown in Figure 2.  $V_{ON}$  is  $-0.5V$  for this part. Find the expression for  $V_{out}$  for this circuit.

**Solution:**

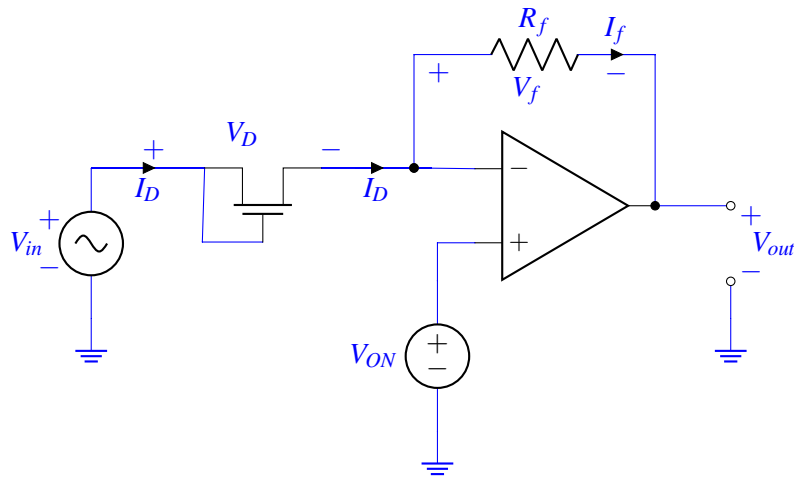


Figure 3: Circuit with an op-amp and the non-linear device, showing  $V_{in}$  and  $V_{out}$

For an ideal op-amp  $I_+ = I_- = 0$ . Hence, using KCL, we can find a relationship between  $I_D$  and  $I_f$ :

$$\begin{aligned} I_f + I_- &= I_D \\ \implies I_f &= I_D \\ \implies I_f &= c_1 V_D^2 + c_2 V_D + c_3 \end{aligned}$$

Now  $V_D$  can be expressed as

$$\begin{aligned} V_D &= V_{in} - (-0.5) \\ \implies V_D &= V_{in} + 0.5 \end{aligned}$$

Since this circuit is in negative feedback,  $u_+ = u_-$ ,  $V_{out}$  can be expressed as

$$\begin{aligned} V_{out} &= -V_f + u_- \\ \implies V_{out} &= -V_f + u_+ \\ \implies V_{out} &= -I_f R_f - 0.5 \\ \implies V_{out} &= -R_f [c_1 V_D^2 + c_2 V_D + c_3] - 0.5 \\ \implies V_{out} &= -R_f [c_1 (V_{in} + 0.5)^2 + c_2 (V_{in} + 0.5) + c_3] - 0.5 \end{aligned}$$

- (b) (4 points) For this part, let us assume that we have a (possibly different) circuit where  $V_{out} = uV_{in}^2 + vV_{in} + w$  and  $u$ ,  $v$  and  $w$  are scalars that are unknown. Tiny measures the output voltage  $V_{out}$  for a few values of  $V_{in}$ .

$V_{in}$	0V	1V	2V	3V
$V_{out}$	$v_{out,0}$	$v_{out,1}$	$v_{out,2}$	$v_{out,3}$

However, it turns out Tiny introduced some noise in the  $V_{out}$  readings during measurement. The readings  $v_{out,0}$ ,  $v_{out,1}$ ,  $v_{out,2}$  and  $v_{out,3}$  are scalars. You utilize the values of  $V_{in}$  and  $V_{out}$  to write a system of

linear equations, which you need to solve for  $u$ ,  $v$  and  $w$ :

$$Q \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \vec{e} = \begin{bmatrix} v_{out,0} \\ v_{out,1} \\ v_{out,2} \\ v_{out,3} \end{bmatrix}$$

Here  $\vec{e}$  is the error introduced due to noise. You use the least squares method to reach the following solution:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -1.0 \\ 0.4 \\ -1.1 \end{bmatrix}$$

Find the projection of  $\begin{bmatrix} v_{out,0} \\ v_{out,1} \\ v_{out,2} \\ v_{out,3} \end{bmatrix}$  on the columnspace of matrix  $Q$ . (Hint: *Start by finding matrix  $Q$ !*)

**Solution:** We can write a system of three linear equations by plugging in the measurement data in  $V_{out} = uV_{in}^2 + vV_{in} + w$ .

$$\begin{aligned} u \cdot 0^2 + v \cdot 0 + w &= v_{out,0} \\ u \cdot 1^2 + v \cdot 1 + w &= v_{out,1} \\ u \cdot 2^2 + v \cdot 2 + w &= v_{out,2} \\ u \cdot 3^2 + v \cdot 3 + w &= v_{out,3} \end{aligned}$$

This system can be written in matrix-vector form:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \approx \begin{bmatrix} v_{out,0} \\ v_{out,1} \\ v_{out,2} \\ v_{out,3} \end{bmatrix}$$

Hence the matrix  $Q$  can be found as following:

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix}$$

The projection of  $\begin{bmatrix} v_{out,0} \\ v_{out,1} \\ v_{out,2} \\ v_{out,3} \end{bmatrix}$  on the columnspace of matrix  $Q$  is given by the product of  $Q$  and the least squares solution, i.e.

$$Q \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1.0 \\ 0.4 \\ -1.1 \end{bmatrix} = \begin{bmatrix} -1.1 \\ -1.7 \\ -4.3 \\ -8.9 \end{bmatrix}$$

(c) (3 points) Now let us assume that the answer for part (b) is given by:  $\begin{bmatrix} \alpha \\ \beta \\ \delta \\ \gamma \end{bmatrix}$ , i.e. the projection of

$\begin{bmatrix} v_{out,0} \\ v_{out,1} \\ v_{out,2} \\ v_{out,3} \end{bmatrix}$  on the columnspace of matrix Q is  $\begin{bmatrix} \alpha \\ \beta \\ \delta \\ \gamma \end{bmatrix}$ . Now Tiny writes a system of linear equations as follows:

$$Q \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \delta \\ \gamma \end{bmatrix}.$$

If you use Gaussian elimination on this system, what will be the result?

- (A) There is no solution for  $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$
- (B) Infinite solutions exist for  $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$
- (C) There is not enough information
- (D) Unique solution exists for  $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$

**Solution:** For any system of linear equation:  $A\vec{x} = \vec{b}$ , a unique solution exists if  $\vec{b}$  belongs in the columnspace of A. The projection of  $\begin{bmatrix} v_{out,0} \\ v_{out,1} \\ v_{out,2} \\ v_{out,3} \end{bmatrix}$  on the columnspace of matrix Q is  $\begin{bmatrix} \alpha \\ \beta \\ \delta \\ \gamma \end{bmatrix}$ , which is in the columnspace of Q. Hence, a unique solution exists for  $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ .



### 11. Denoising with Orthogonal Matching Pursuit (21 points)

Suppose we have a noisy measurement  $\vec{y}_{\text{noisy}}$  such that

$$\vec{y}_{\text{noisy}} = \vec{y}_{\text{clean}} + \vec{\eta},$$

where  $\vec{y}_{\text{clean}}$  is the measurement with no noise and  $\vec{\eta}$  is an unknown vector of noise. We'd like to remove the noise from  $\vec{y}_{\text{noisy}}$ , but this is in general a hard problem because we don't know  $\vec{y}_{\text{clean}}$  or  $\vec{\eta}$  so there are infinite possible pairs that satisfy the above equation.

Suppose we additionally know that

$$\vec{y}_{\text{clean}} = \mathbf{A}\vec{x}$$

where  $\vec{x}$  is a sparse vector and  $\mathbf{A}$  is an  $m \times 4$  matrix with columns  $\vec{a}_1 \dots \vec{a}_4$ . You go to your favorite TA Moses and he suggests you can use OMP to determine  $\vec{x}$  and find  $\vec{y}_{\text{clean}}$ !

He gives you the following table of inner products:

$\langle \cdot, \cdot \rangle$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{\eta}$
$\vec{a}_1$	1	-0.2	0.1	0	-0.2
$\vec{a}_2$		1	0.2	0.3	0
$\vec{a}_3$			1	0.3	-0.2
$\vec{a}_4$				1	$\gamma$

For example, the inner product  $\langle \vec{a}_1, \vec{a}_2 \rangle = -0.2$  and  $\langle \vec{a}_1, \vec{a}_3 \rangle = 0.1$ .

(a) (3 points) Select all of the following that are guaranteed to be true based on the table above.

- (1) If  $\gamma = 1$ , then  $\vec{\eta} = \vec{a}_4$
- (2)  $\|\vec{a}_3\| = \|\vec{a}_2\|$
- (3)  $\text{proj}_{\vec{a}_2}(\vec{\eta}) = \vec{0}$
- (4)  $\|\vec{\eta}\| = 1$
- (5)  $\vec{a}_1$  and  $\vec{a}_2$  are orthogonal

**Solution:** We start with statement (2):

$$\|\vec{a}_3\| = \sqrt{\langle \vec{a}_3, \vec{a}_3 \rangle} = \sqrt{1} = 1.$$

Similarly,  $\|\vec{a}_2\| = \sqrt{\langle \vec{a}_2, \vec{a}_2 \rangle} = \sqrt{1} = 1$ . Therefore,  $\|\vec{a}_3\| = \|\vec{a}_2\|$

Therefore, statement (2) is true.

Next, statement (3):

$$\text{proj}_{\vec{a}_2}(\vec{\eta}) = \frac{\langle \vec{a}_2, \vec{\eta} \rangle}{\langle \vec{a}_2, \vec{a}_2 \rangle} \vec{a}_2 = 0 \vec{a}_2 = \vec{0}.$$

Therefore, statement (3) is true.

Statements (4) and (1):

We don't know the norm of  $\vec{\eta}$  so can't tell if (4) is true from the table. If we knew that  $\|\vec{\eta}\|$  were 1, then (1) would be true, but we don't know that's the case, so (1) and (4) are not correct.

Statement (5):

From the table,  $\langle \vec{a}_1, \vec{a}_2 \rangle = -0.2$  so  $\vec{a}_1$  and  $\vec{a}_2$  are not orthogonal.

- (b) (6 points) Moses reminds you that the first step of OMP is to calculate the inner product of your measurement vector with each column of  $\mathbf{A}$ .

Let  $\vec{x} = [0 \ 0 \ 8 \ 0]^T$ . Calculate  $\langle \vec{a}_1, \vec{y}_{\text{noisy}} \rangle$ . Show your work. (*Hint: Find an expression for the unknown inner product in terms of  $\vec{x}$  and the inner products in the table.*)

**Solution:** Let the elements of  $\vec{x}$  be  $x_1 \dots x_4$ .

$$\begin{aligned} \langle \vec{a}_1, \vec{y}_{\text{noisy}} \rangle &= \langle \vec{a}_1, \vec{y}_{\text{clean}} + \vec{\eta} \rangle \\ &= \langle \vec{a}_1, \mathbf{A}\vec{x} + \vec{\eta} \rangle \\ &= \langle \vec{a}_1, \mathbf{A}\vec{x} \rangle + \langle \vec{a}_1, \vec{\eta} \rangle \\ &= \langle \vec{a}_1, \sum_{i=1}^4 x_i \vec{a}_i \rangle + \langle \vec{a}_1, \vec{\eta} \rangle \\ &= \sum_{i=1}^4 x_i \langle \vec{a}_1, \vec{a}_i \rangle + \langle \vec{a}_1, \vec{\eta} \rangle \end{aligned}$$

All the  $x_i$  are zero except  $x_3$ , so the summation can be simplified to

$$= x_3 \langle \vec{a}_1, \vec{a}_3 \rangle + \langle \vec{a}_1, \vec{\eta} \rangle$$

Now we plug in the numbers from the table:

$$\begin{aligned} &= 8(0.1) + (-0.2) \\ &= 0.6 \end{aligned}$$

- (c) (2 points) Moses helps you calculate the remaining inner products:

$$\langle \vec{a}_2, \vec{y}_{\text{noisy}} \rangle = 1.6 \qquad \langle \vec{a}_3, \vec{y}_{\text{noisy}} \rangle = 7.8 \qquad \langle \vec{a}_4, \vec{y}_{\text{noisy}} \rangle = \gamma + 2.4$$

Find the range of values of  $\gamma$  for which the first iteration of OMP will select 3 as the index of the nonzero entry of  $\vec{x}$ . (*Note: the first entry in  $\vec{x}$  has index "1" and the last entry has index "4."*)

**Solution:** OMP selects the largest absolute value, so to correctly select index 3,  $\langle \vec{a}_2, \vec{y}_{\text{noisy}} \rangle$  must be largest. Therefore,  $|\langle \vec{a}_4, \vec{y}_{\text{noisy}} \rangle| < |\langle \vec{a}_3, \vec{y}_{\text{noisy}} \rangle|$  so  $|\gamma + 2.4| < 5.1$ . This is equivalent to  $-10.2 < \gamma < 5.4$ .

- (d) (3 points) Now assume we have a new noisy measurement with different  $\vec{y}_{\text{clean}}$ ,  $\vec{y}_{\text{noisy}}$ , and  $\vec{x}$  than the prior parts. The matrix  $\mathbf{A}$  is unchanged and the table is still valid. We repeat the table here for your convenience:

$\langle \cdot, \cdot \rangle$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{\eta}$
$\vec{a}_1$	1	-0.2	0.1	0	-0.2
$\vec{a}_2$		1	0.2	0.3	0
$\vec{a}_3$			1	0.3	-0.2
$\vec{a}_4$				1	$\gamma$

Using the new noisy measurement, you run two iterations of OMP and get the following calculations:

- Iteration 1: You find index  $i = 2$  maximizes  $|\langle \vec{e}, \vec{a}_i \rangle|$  and you calculate  $\langle \vec{a}_2, \vec{y}_{\text{noisy}} \rangle = 6$ .
- Iteration 2: You find index  $i = 1$  maximizes  $|\langle \vec{e}, \vec{a}_i \rangle|$  and you calculate  $\langle \vec{a}_1, \vec{y}_{\text{noisy}} \rangle = -9$ .

What is  $\hat{\vec{x}}$ , the OMP estimate of  $\vec{x}$ , after these two iterations? Let  $\hat{\vec{x}} = [\hat{x}_1 \ \hat{x}_2 \ \hat{x}_3 \ \hat{x}_4]^T$ .

**Solution:** We form a new matrix  $\mathbf{B}$  containing the columns of  $\mathbf{A}$  corresponding to the indices that maximized the inner products.

$$\mathbf{B} = \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix}$$

Then we calculate the estimate of the non-zero values of  $\hat{\vec{x}}$  with least squares:

$$\begin{aligned} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} &= (\mathbf{B}^T \mathbf{B})^{-1} \mathbf{B}^T \vec{y}_{\text{noisy}} \\ &= \left( \begin{bmatrix} - & \vec{a}_1^T & - \\ - & \vec{a}_2^T & - \end{bmatrix} \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} \right)^{-1} \begin{bmatrix} - & \vec{a}_1^T & - \\ - & \vec{a}_2^T & - \end{bmatrix} \begin{bmatrix} | & | \\ \vec{y}_{\text{noisy}} \\ | & | \end{bmatrix} \\ &= \begin{bmatrix} \langle \vec{a}_1, \vec{a}_1 \rangle & \langle \vec{a}_1, \vec{a}_2 \rangle \\ \langle \vec{a}_1, \vec{a}_2 \rangle & \langle \vec{a}_2, \vec{a}_2 \rangle \end{bmatrix}^{-1} \begin{bmatrix} \langle \vec{a}_1, \vec{y}_{\text{noisy}} \rangle \\ \langle \vec{a}_2, \vec{y}_{\text{noisy}} \rangle \end{bmatrix} \\ \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} &= \begin{bmatrix} 1 & -0.2 \\ -0.2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -9 \\ 6 \end{bmatrix} \end{aligned}$$

We know that the remaining elements of  $\hat{\vec{x}}$  are all zero:

$$\begin{bmatrix} \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- (e) (3 points) Let  $\hat{\mathbf{y}} = \mathbf{A}\hat{\mathbf{x}}$  be the best estimate of  $\vec{y}_{\text{noisy}}$  after the two iterations of OMP described in part (d). (Note,  $\hat{\mathbf{y}}$  is also your estimate of  $\vec{y}_{\text{clean}}$ .) You want to run a third iteration of OMP, so you calculate the error,  $\vec{e} = \vec{y}_{\text{noisy}} - \hat{\mathbf{y}}$ .

What is the inner product of  $\vec{e}$  with each of the four columns,  $\vec{a}_1 \dots \vec{a}_4$ ?

- (A)  $\langle \vec{a}_i, \vec{e} \rangle = 0$  for  $i = 1, 2$ . There isn't enough information to calculate  $\langle \vec{a}_i, \vec{e} \rangle$  for  $i = 3, 4$ .  
 (B) There isn't enough information to calculate  $\langle \vec{a}_i, \vec{e} \rangle$  for any  $i$ .  
 (C)  $\langle \vec{a}_i, \vec{e} \rangle = 0$  for all  $i$ .  
 (D)  $\langle \vec{a}_1, \vec{e} \rangle = -0.2$ ,  $\langle \vec{a}_2, \vec{e} \rangle = 0$ ,  $\langle \vec{a}_3, \vec{e} \rangle = -0.2$ ,  $\langle \vec{a}_4, \vec{e} \rangle = \gamma$   
 (E)  $\langle \vec{a}_i, \vec{e} \rangle = 0$  for  $i = 3, 4$ . There isn't enough information to calculate  $\langle \vec{a}_i, \vec{e} \rangle$  for  $i = 1, 2$ .

**Solution:** Since we solved a least squares problem in part, the error is orthogonal to all of the columns we selected in a prior iteration of OMP. Since the first two iterations found columns 2 and 1, these inner products will be zero.

We can't calculate the inner product with  $\vec{a}_3$  or  $\vec{a}_4$  since this would require knowing  $\langle \vec{a}_3, \vec{y}_{\text{noisy}} \rangle$  and  $\langle \vec{a}_4, \vec{y}_{\text{noisy}} \rangle$ , respectively, and these quantities were not given.

- (f) (4 points) Now, let  $\mathbf{B} \in \mathbb{R}^{9 \times 10}$ . We use OMP to solve

$$\vec{v} = \mathbf{B}\vec{u}.$$

After 8 iterations of OMP, we get  $\hat{\vec{u}}$  as our best estimate of  $\vec{u}$ , and we find that the error  $\vec{v} - \mathbf{B}\hat{\vec{u}} = \vec{0}$ . Which of the following statements must be true? Choose all that apply. Assume  $\vec{v} \neq \vec{0}$ .

- (1) There is at least one column of  $\mathbf{B}$  that is orthogonal to  $\vec{v}$ .  
 (2)  $\text{rank}(\mathbf{B}) \leq 8$   
 (3) There is at least one column of  $\mathbf{B}$  that is not orthogonal to  $\vec{v}$ .  
 (4)  $\vec{v} \in C(\mathbf{B})$ , where  $C(\mathbf{B})$  is the columnspace of  $\mathbf{B}$ .  
 (5)  $\text{rank}(\mathbf{B}) = 8$   
 (6)  $\text{rank}(\mathbf{B}) \geq 8$

**Solution:** Since we get an error of zero after 8 iterations,  $\vec{v}$  is a linear combination of 8 linearly independent columns of  $\mathbf{B}$ . This means that  $\vec{v}$  is in the columnspace of  $\mathbf{B}$ . (So option (4) is true.) This also means that  $\vec{v}$  is not orthogonal to at least 8 columns of  $\mathbf{B}$ . (Option (3) is true.) Now  $\mathbf{B}$  has 10 column vectors. For the other two column vectors of  $\mathbf{B}$ , which do not make up  $\vec{v}$ , two scenarios are possible:

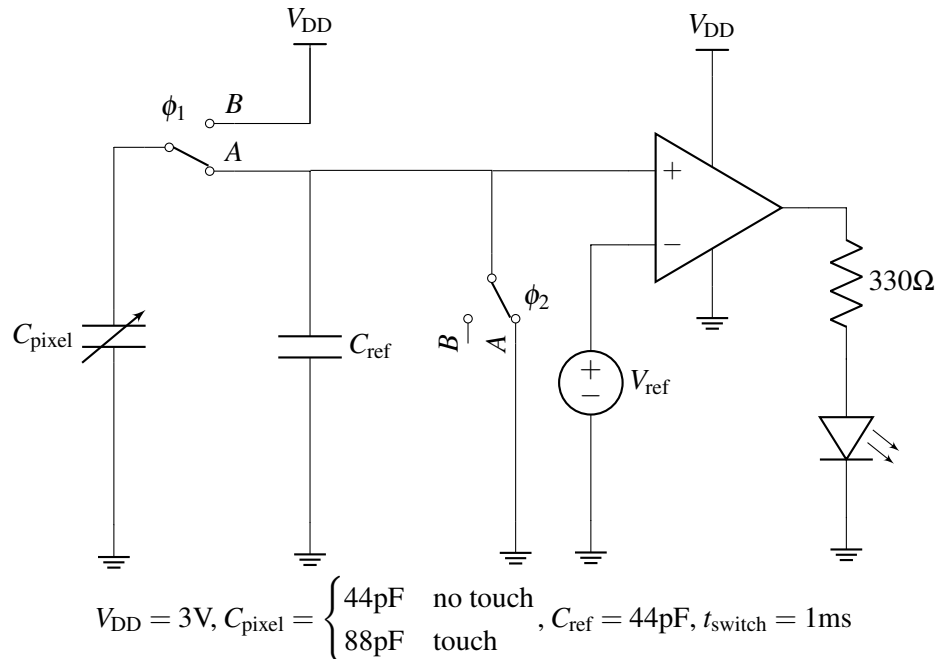
- (1) For a matrix  $\in \mathbb{R}^{9 \times 10}$ , at most 9 columns can be linearly independent. In this scenario, there are 9 linearly dependent columns. So the rank of the  $\mathbf{B}$  would be 9. The ninth linearly independent column does not take part to make up  $\vec{v}$ , which means its projection on  $\vec{v}$  is 0, i.e. it must be orthogonal to  $\vec{v}$ .  
 (2) Both the ninth and tenth columns are linearly dependent on the other 8 columns. In this case, the rank of the  $\mathbf{B}$  will be 8.

Since either of these two scenarios are possible we know that rank of the  $\mathbf{B}$  will be  $\geq 8$  (Option (6) is true). However, we cannot be sure if there is a ninth linearly independent column, i.e. we don't know there is a column of  $\mathbf{B}$  that is orthogonal to  $\vec{v}$ , i.e. option (1) does not need to be true.

### 12. Capacitive Touchscreen 2: Electric Boogaloo (15 points)

As an expert on the capacitive touchscreen, you have been recruited to redesign the circuit from lab to be used in a professional setting.

- (a) (3 points) At first, you revisit the capacitive touchscreen circuit below, in which  $C_{\text{pixel}}$  varies depending on whether or not a touch has occurred, and the switches  $\phi_1$ ,  $\phi_2$  are sequentially configured to create three different states: the **clean** state, during which  $C_{\text{ref}}$  is reset to 0, the **charge** state, during which  $C_{\text{pixel}}$  is charged up, and the **read** state, during which the charge is being redistributed (shared), and read out. These states are cyclically repeated at a certain rate, providing a constantly updated reading of whether or not a touch has been detected.



Which of the following demonstrates a correct switching order? (i.e. a switching order that can lead to detecting a touch by properly cycling through the clean, charge, and read states).

Each step will be notated as follows:  $(\phi_1 = [A \text{ or } B], \phi_2 = [A \text{ or } B])$  where A and B correspond to the labeled switch connections. Only one switch can be modified per step. Select all that apply.

- (1)  $(\phi_1 = A, \phi_2 = A), (\phi_1 = A, \phi_2 = B), (\phi_1 = B, \phi_2 = B), (\phi_1 = A, \phi_2 = B)$
- (2)  $(\phi_1 = A, \phi_2 = A), (\phi_1 = A, \phi_2 = B), (\phi_1 = A, \phi_2 = A), (\phi_1 = A, \phi_2 = A)$
- (3)  $(\phi_1 = A, \phi_2 = A), (\phi_1 = A, \phi_2 = B), (\phi_1 = B, \phi_2 = B), (\phi_1 = B, \phi_2 = A)$
- (4)  $(\phi_1 = A, \phi_2 = A), (\phi_1 = B, \phi_2 = A), (\phi_1 = B, \phi_2 = B), (\phi_1 = A, \phi_2 = B)$
- (5)  $(\phi_1 = A, \phi_2 = A), (\phi_1 = B, \phi_2 = A), (\phi_1 = A, \phi_2 = A), (\phi_1 = A, \phi_2 = B)$

**Solution:** All of the options start at  $(\phi_1 = A, \phi_2 = A)$  - this stage is the cleaning step, where the Op Amp non-inverting input,  $C_{\text{pixel}}$ , and  $C_{\text{ref}}$  are all connected to ground, discharging all built up charges in the node.

After the cleaning step, the clean switch,  $\phi_2$  must be disconnected from ground, resulting in  $(\phi_1 = A, \phi_2 = B)$ . We can then charge up  $C_{\text{pixel}}$  to  $V_{DD}$ , thus the next state is  $(\phi_1 = B, \phi_2 = B)$ .

Alternatively, we could first charge up the capacitor  $C_{\text{pixel}}$ , going from state  $(\phi_1 = A, \phi_2 = A)$  to state  $(\phi_1 = B, \phi_2 = A)$   $V_{DD}$  and then disconnect the clean switch from ground to ensure that charge will not be lost when we turn  $\phi_1$  back to A.

The important thing is to have state  $(\phi_1 = B, \phi_2 = B)$  as second to last.

Lastly, we would like to share the charge built up on  $C_{\text{pixel}}$  with  $C_{\text{ref}}$ , so we connect  $\phi_1$  back to the Op Amp non-inverting node, finishing with  $(\phi_1 = A, \phi_2 = B)$

Summing it up the two correct options are:

- (1)  $(\phi_1 = A, \phi_2 = A), (\phi_1 = A, \phi_2 = B), (\phi_1 = B, \phi_2 = B), (\phi_1 = A, \phi_2 = B)$   
 (4)  $(\phi_1 = A, \phi_2 = A), (\phi_1 = B, \phi_2 = A), (\phi_1 = B, \phi_2 = B), (\phi_1 = A, \phi_2 = B)$

- (b) (3 points) What is the amount of charge,  $Q_{\text{no-touch}}$ , that  $V_{\text{DD}}$  has to provide in every clean-charge-read cycle of operation, assuming there is **no touch**? Is the amount of charge,  $Q_{\text{touch}}$ , any different when **there is a touch**?

*Hint:* Do not add up the amount of charge in each step, think about conservation of charge and what that means in the context of calculating total charge in a cycle.

**Note:** Do not consider the charge required for the comparator to work in your calculations.

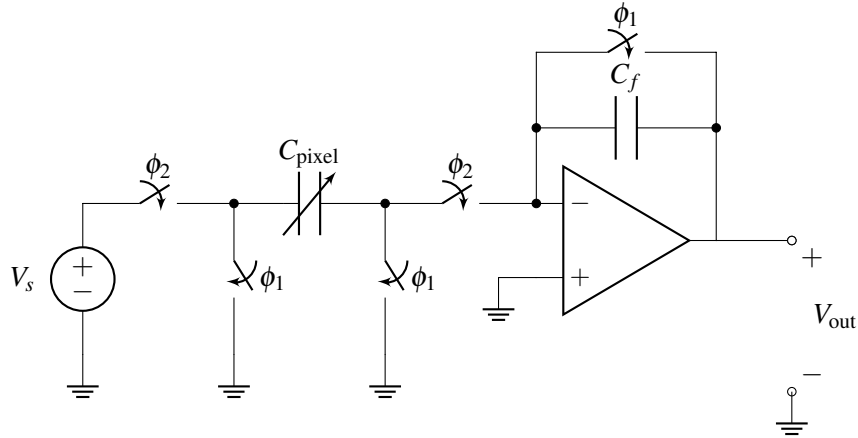
- (A)  $Q_{\text{no-touch}} = 132\text{pC}$ , No,  $Q_{\text{touch}} = Q_{\text{no-touch}}$   
 (B)  $Q_{\text{no-touch}} = 264\text{pC}$ , No,  $Q_{\text{touch}} = Q_{\text{no-touch}}$   
 (C)  $Q_{\text{no-touch}} = 264\text{pC}$ , Yes:  $Q_{\text{touch}} = \frac{Q_{\text{no-touch}}}{2}$   
 (D)  $Q_{\text{no-touch}} = 66\text{pC}$ , Yes:  $Q_{\text{touch}} = 2Q_{\text{no-touch}}$   
 (E)  $Q_{\text{no-touch}} = 66\text{pC}$ , Yes:  $Q_{\text{touch}} = \frac{Q_{\text{no-touch}}}{2}$   
 (F)  $Q_{\text{no-touch}} = 132\text{pC}$ , Yes:  $Q_{\text{touch}} = 2Q_{\text{no-touch}}$

**Solution:** The key thing to realize here, is that the total amount of charge available in the system comes from the charge phase, specifically  $(\phi_1 = B, \phi_2 = B)$  (or  $(\phi_1 = B, \phi_2 = A)$ , depending on the switching order). The amount of charge on a charged up capacitor is given by the equation  $Q = CV$ ; thus the total amount of charge would be  $Q_{\text{no-touch}} = C_{\text{pixel}}V_{\text{DD}} = 132\text{pC}$ .

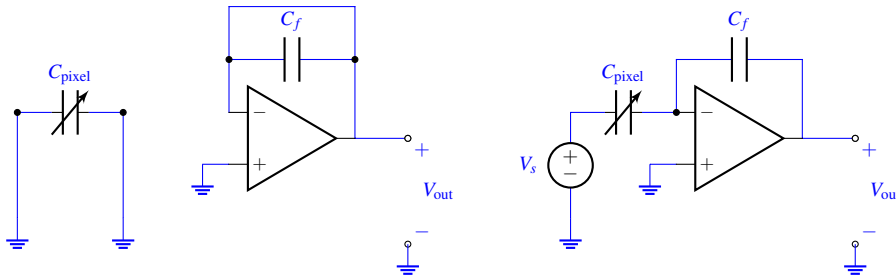
Also, since the total charge depends on the value of  $C_{\text{pixel}}$ , it will double when a touch occurs, i.e.  $Q_{\text{touch}} = 2Q_{\text{no-touch}}$ .

Hence, the correct option is (F).

- (c) (6 points) After analyzing the circuit used in your lab you are given a new design to look at. This design has the property that the output is not a binary function of touch vs no-touch, but a linear function of the variable capacitance  $C_{\text{pixel}}$ . It operates in 2 phases: phase 1, where switches  $\phi_1$  are closed and switches  $\phi_2$  are open, and phase 2, where switches  $\phi_2$  are closed and switches  $\phi_1$  are open. Calculate  $V_{\text{out}}$  during **phase 2**, as a function of the voltage source,  $V_s$ , and the capacitors,  $C_{\text{pixel}}$ ,  $C_f$ . **Show your work.**



**Solution:** Redrawing the circuit in both phases we get:



With the equivalent circuit during Phase 1 shown on the left and the equivalent circuit during Phase 2 shown on the right. We can see that the floating node during phase 2 is the inverting terminal of the op amp.

Calculating the charge on the right plate of  $C_{\text{pixel}}$  and the left plate of  $C_f$  during phases 1 and 2 we get:

$$Q_{u^-}^{\phi_1} = 0$$

$$Q_{u^-}^{\phi_2} = -(V_s - u^-) C_{\text{pixel}} + (u^- - V_{\text{out}}) C_f$$

Finally, equating the two we get that:

$$-(V_s - u^-) C_{\text{pixel}} + (u^- - V_{\text{out}}) C_f = 0$$

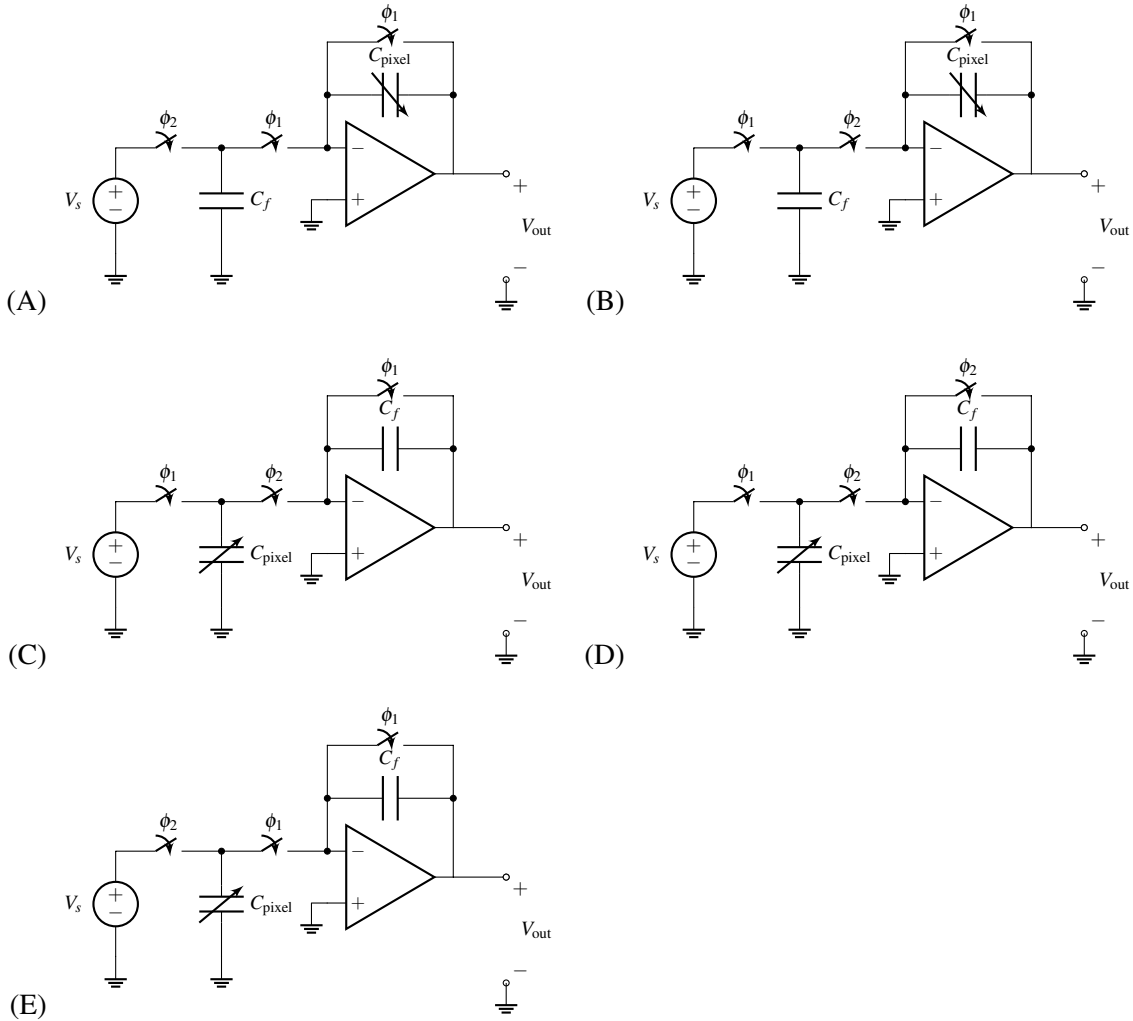
$$\Rightarrow -V_s C_{\text{pixel}} - V_{\text{out}} C_f = 0$$

$$\Rightarrow V_{\text{out}} = -\frac{C_{\text{pixel}}}{C_f} V_s$$

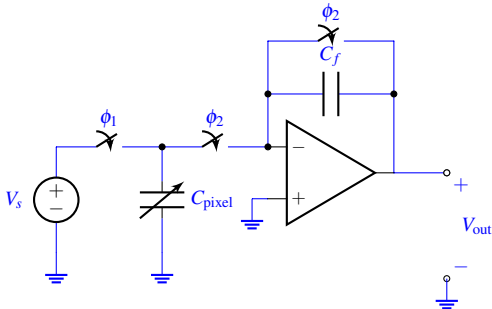
where we have used the fact that because of Golden Rule #2:  $u^- = u^+ = 0$ .

We see that indeed  $V_{\text{out}}$  is a linear function of the variable capacitance  $C_{\text{pixel}}$ .

(d) (3 points) Your mentor points out that the circuit from part (c) can be further simplified and two of the switches can be removed while maintaining the same functionality. Which of the following circuits achieves this goal?



**Solution:** The correct answer is:



We see that for this circuit as well the floating node during phase 2 is the inverting terminal of the opamp so we will identify the charge on the **top plate of  $C_{pixel}$  and the left plate of  $C_f$**  during both phases:



$$Q_{u^-}^{\phi_1} = C_{\text{pixel}} V_s$$
$$Q_{u^-}^{\phi_2} = (u^- - V_{\text{out}}) C_f$$

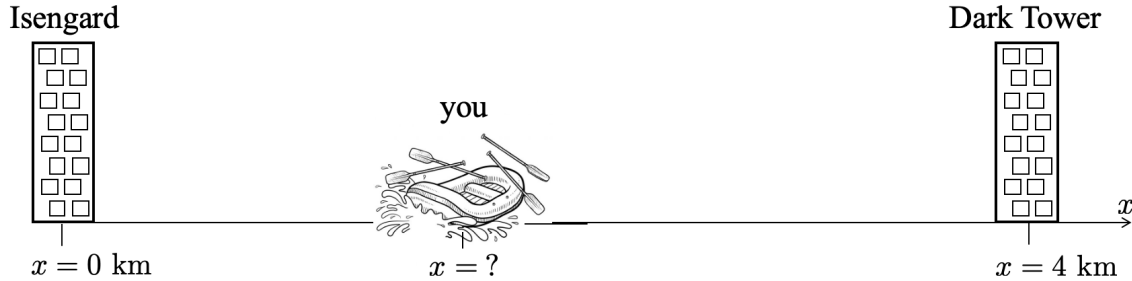
Equating the two and applying GR #2 ( $u^- = u^+ = 0$ ) we get:

$$C_{\text{pixel}} V_s = (u^- - V_{\text{out}}) C_f$$
$$\Rightarrow V_{\text{out}} = -\frac{C_{\text{pixel}}}{C_f} V_s$$

Which is the same as in part (c) but with 2 less switches.

**13. One does not simply raft into Mordor (23 points)**

You've decided to go rafting to celebrate the end of your summer classes! Unfortunately, an hour into your trip you realize that there are no landmarks that look familiar, so you're not sure how far you are from your starting point. You remember from your studies of the area that there are two towers, Isengard at the position  $x = 0$  km and the Dark Tower at  $x = 4$  km. You know you are between the two towers, as shown below:



You know that each tower emits a sound signal once a day at midday. Specifically, Isengard will emit  $\vec{b}_1$  and the Dark Tower will emit  $\vec{b}_2$ :

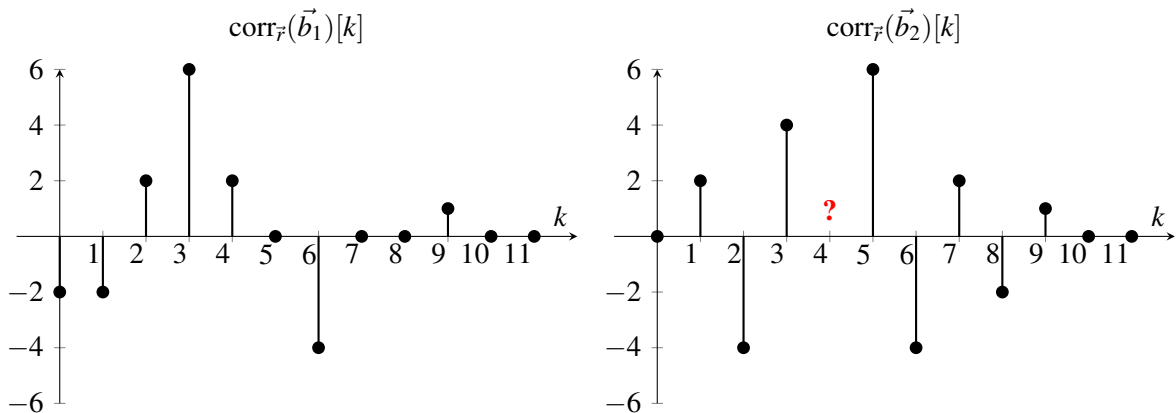
$$\vec{b}_1 = [-1 \quad -1 \quad -1 \quad 1 \quad 1]^T \quad \vec{b}_2 = [-1 \quad 1 \quad -1 \quad 1 \quad -1]^T$$

Both signals are emitted at a rate of 2 samples per seconds (i.e. sample interval is 0.5 sec), and the signals are emitted only once.

It's only a few minutes from midday so you decide to wait. You use an app on your phone to record the incoming signal (the app also records at 2 samples per second). You start recording at exactly 12:00 PM and receive the following:

$$\vec{r} = [0 \quad 0 \quad 0 \quad -1 \quad -1 \quad -2 \quad 2 \quad 0 \quad 1 \quad -1 \quad 0 \quad 0]^T$$

- (a) (2 points) Your first step is to calculate a linear cross-correlation between  $\vec{r}$  and each known tower signature. The cross-correlations are plotted below. Calculate the missing value, which is denoted with a question mark.



**Solution:** To fill in the missing cross-correlation values, observe that the missing value is in  $\text{corr}_{\vec{r}}(\vec{b}_2)$  is at index 4. Therefore, we must calculate the inner product between  $\vec{r}$  and a version of  $\vec{b}_2$  that is shifted 4 samples to the right.

$$\vec{r} = [0 \quad 0 \quad 0 \quad -1 \quad -1 \quad -2 \quad 2 \quad 0 \quad 1 \quad -1 \quad 0 \quad 0]$$

$$\vec{b}_{2,\text{shifted by 4}} = [0 \quad 0 \quad 0 \quad 0 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 0 \quad 0 \quad 0]$$

We calculate:

$$\begin{aligned}\text{corr}_{\vec{r}}(\vec{b}_2)[4] &= (-1)(0) + (-1)(-1) + (-2)(1) + (2)(-1) + (0)(1) + (1)(-1) + (-1)(0) \\ &= 1 - 2 - 2 + 0 - 1 \\ &= -4\end{aligned}$$

- (b) (3 points) Recall that the signals were emitted from each building at the same time (12:00 PM). How many seconds after 12:00 PM did it take for the signal from Isengard to reach you? What about the signal from the Dark Tower? Assume that environmental noise (besides the tower-emitted signals) is minimal.

**Solution:** From the completed correlation diagrams solved for in (a), we find that  $\text{corr}_{\vec{r}}(\vec{b}_1)$  is maximized at sample 3 (value 6) and  $\text{corr}_{\vec{r}}(\vec{b}_2)$  is maximized at sample 5 (value 6). The time needed for the app to measure 3 samples is  $\frac{3 \text{ samples}}{2 \text{ samples / sec}} = 1.5 \text{ sec}$ . Similarly, the time needed for the app to measure 5 samples is  $\frac{5 \text{ samples}}{2 \text{ samples / sec}} = 2.5 \text{ sec}$ . Hence, the number of seconds it took for the signal from Isengard to reach you was 1.5 sec. From the Dark Tower, it took 2.5 sec seconds.

- (c) (3 points) Now, assume that you received Isengard's signal 8 seconds after it was sent and you received the Dark Tower's signal 2 seconds after it was sent. Can you determine your exact position  $x$ ? If yes, calculate your position. If not, explain why not. Assume sound travels at 340 m/s.

**Solution:** Sound travels at 340 m/s so the measurement from Isengard tells you that you are  $8 \text{ sec} \times 340 \text{ m/sec} = 2,720 \text{ m} = 2.72 \text{ km}$  from Isengard.

The measurement from the Dark Tower tells you that you are  $2 \text{ sec} \times 340 \text{ m/sec} = 680 \text{ m} = 0.68 \text{ km}$  from Isengard.

You know that the two towers are 4 km apart. Therefore the measurement from Isengard indicates that you are at  $x = 2.72 \text{ km}$  and the measurement from the Dark Tower indicates that you are at  $x = 4 \text{ km} - 0.68 \text{ km} = 3.32 \text{ km}$ . These measurements are inconsistent, so you cannot determine your exact position.

(d) (3 points) You see a giant eagle, so you get out of your raft to follow it. But you soon realize that you don't know your  $x$  position *or* your  $y$  position! Luckily, you have a phone app which tells you that are:

- $d_1$  km away from Isengard which is located at  $x = 0$  km,  $y = 0$  km
- $d_2$  km away from the Dark Tower which is located at  $x = 4$  km,  $y = 0$  km
- $d_3$  km away from Minas Tirith which is located at  $x = 1$  km,  $y = 3$  km

Write a system of linear equations of the form  $\mathbf{A}\vec{x} = \vec{b}$  that you can solve to find your position.

Let  $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$  where  $x, y$  have units of kilometers (km).

**Solution:**

Let  $\vec{a}_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$  be the position of the Dark Tower and  $\vec{a}_3 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$  be the position of Minas Tirith.

The three towers give us the following equations:

- Isengard:  $\|\vec{x}\|^2 = d_1^2$
- The Dark Tower:  $\|\vec{x} - \vec{a}_2\|^2 = d_2^2$
- Minas Tirith:  $\|\vec{x} - \vec{a}_3\|^2 = d_3^2$

Rewriting these using transpose notation we get:

$$\vec{x}^T \vec{x} = d_1^2 \quad (4)$$

$$\vec{x}^T \vec{x} - 2\vec{a}_2^T \vec{x} + \|\vec{a}_2\|^2 = d_2^2 \quad (5)$$

$$\vec{x}^T \vec{x} - 2\vec{a}_3^T \vec{x} + \|\vec{a}_3\|^2 = d_3^2 \quad (6)$$

We subtract equation 2 from equation 1, and separately we subtract equation 3 from equation 1. Then we get:

$$2\vec{a}_2^T \vec{x} - \|\vec{a}_2\|^2 = d_1^2 - d_2^2$$

and

$$2\vec{a}_3^T \vec{x} - \|\vec{a}_3\|^2 = d_1^2 - d_3^2$$

These two equations are linear in  $\vec{x}$ , write them in matrix-vector form:

$$\begin{bmatrix} 2\vec{a}_2^T \\ 2\vec{a}_3^T \end{bmatrix} \vec{x} = \begin{bmatrix} \|\vec{a}_2\|^2 + d_1^2 - d_2^2 \\ \|\vec{a}_3\|^2 + d_1^2 - d_3^2 \end{bmatrix}.$$

We see that

$$\mathbf{A} = \begin{bmatrix} 2\vec{a}_2^T \\ 2\vec{a}_3^T \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} \|\vec{a}_2\|^2 + d_1^2 - d_2^2 \\ \|\vec{a}_3\|^2 + d_1^2 - d_3^2 \end{bmatrix}.$$

Plugging in the values of  $\vec{a}_2$  and  $\vec{a}_3$  gives:

$$\mathbf{A} = \begin{bmatrix} 8 & 0 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 16 + d_1^2 - d_2^2 \\ 10 + d_1^2 - d_3^2 \end{bmatrix}.$$

Unfortunately, your app hasn't been updated in a while, and there is a bug in the app. Rather than telling you the true  $\vec{b}$ , the app gives you  $\vec{b}_0 = \vec{b} + \vec{e}$  where  $\vec{e}$  is the error from the bug. Nevertheless, you attempt to find your position by solving

$$\mathbf{A}\vec{x} = \vec{b}_0,$$

where  $\vec{x} = [x \ y]^T$  as before. Suppose the matrix  $\mathbf{A}$  has two unique non-zero eigenvalues  $\lambda_1, \lambda_2$  with associated eigenvectors,  $\vec{v}_1, \vec{v}_2$ , where both eigenvectors are normalized. You can write the error vector,  $\vec{e}$ , as a linear combination of the eigenvectors of  $\mathbf{A}$  as follows:

$$\vec{e} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2$$

(e) (8 points) Show that

$$\Delta = \left\| \frac{\alpha_1}{\lambda_1} \vec{v}_1 + \frac{\alpha_2}{\lambda_2} \vec{v}_2 \right\|,$$

where  $\Delta$  is the distance between your calculated position ( $\vec{x}$  found using  $\vec{b}_0$ ) and your true position ( $\vec{x}$  found using  $\vec{b}$ ). Clearly justify each step.

**Solution:** Let  $\vec{x}_{\text{calc}}$  be your calculated position and let  $\vec{x}_{\text{true}}$  be your true position.

$$\mathbf{A}\vec{x}_{\text{true}} = \vec{b}$$

$$\mathbf{A}\vec{x}_{\text{calc}} = \vec{b}_0 = \vec{b} + \vec{e}$$

We know that  $\mathbf{A}$  is invertible because we are told that  $\mathbf{A}$  has two non-zero eigenvalues.

$$\vec{x}_{\text{true}} = \mathbf{A}^{-1}\vec{b}$$

$$\vec{x}_{\text{calc}} = \mathbf{A}^{-1}\vec{b}_0 = \mathbf{A}^{-1}\vec{b} + \mathbf{A}^{-1}\vec{e}$$

We're interested in the difference between the calculated and true positions:

$$\vec{x}_{\text{calc}} - \vec{x}_{\text{true}} = \mathbf{A}^{-1}\vec{b} - (\mathbf{A}^{-1}\vec{b} + \mathbf{A}^{-1}\vec{e})$$

$$\vec{x}_{\text{calc}} - \vec{x}_{\text{true}} = \mathbf{A}^{-1}\vec{e}$$

We plug in the expression for  $\vec{e}$  in terms of the eigenvectors of  $\mathbf{A}$ :

$$\vec{x}_{\text{calc}} - \vec{x}_{\text{true}} = \mathbf{A}^{-1}(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2)$$

$$\vec{x}_{\text{calc}} - \vec{x}_{\text{true}} = \alpha_1 \mathbf{A}^{-1}\vec{v}_1 + \alpha_2 \mathbf{A}^{-1}\vec{v}_2$$

We know that  $\vec{v}_1, \vec{v}_2$  are eigenvectors of  $\mathbf{A}$  and  $\mathbf{A}$  is invertible:

$$\mathbf{A}\vec{v}_i = \lambda_i \vec{v}_i$$

$$\vec{v}_i = \lambda_i \mathbf{A}^{-1}\vec{v}_i$$

$$\frac{1}{\lambda_i} \vec{v}_i = \mathbf{A}^{-1}\vec{v}_i$$

Therefore,  $\vec{v}_1, \vec{v}_2$  are also eigenvectors of  $\mathbf{A}^{-1}$  with associated eigenvalues  $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}$ . We plug this into the expression for  $\vec{x}_{\text{calc}} - \vec{x}_{\text{true}}$ :

$$\vec{x}_{\text{calc}} - \vec{x}_{\text{true}} = \alpha_1 \frac{1}{\lambda_1} \vec{v}_1 + \alpha_2 \frac{1}{\lambda_2} \vec{v}_2$$

The distance between the calculated and true positions is the norm of the difference vector. Therefore:

$$\Delta = \|\vec{x}_{\text{calc}} - \vec{x}_{\text{true}}\| = \left\| \frac{\alpha_1}{\lambda_1} \vec{v}_1 + \frac{\alpha_2}{\lambda_2} \vec{v}_2 \right\|$$

- (f) (4 points) Let  $\lambda_1 = 2$ ,  $\lambda_2 = 1$ ,  $\alpha_1 = 4$ , and  $\alpha_2 = -1$ . Find an upper bound on the distance  $\Delta$ . In other words, find a constant,  $c$ , such that  $\Delta$  is always less than  $c$ , regardless of the values of  $\vec{v}_1, \vec{v}_2$ . Choose the smallest value of  $c$  such that the expression  $\Delta \leq c$  is always true. (Your answer should not depend on  $\vec{v}_1, \vec{v}_2$ .)

**Solution:**

$$\begin{aligned} \Delta &= \left\| \frac{\alpha_1}{\lambda_1} \vec{v}_1 + \frac{\alpha_2}{\lambda_2} \vec{v}_2 \right\| \\ \Delta &\leq \left\| \frac{\alpha_1}{\lambda_1} \vec{v}_1 \right\| + \left\| \frac{\alpha_2}{\lambda_2} \vec{v}_2 \right\| \\ \Delta &\leq \left| \frac{\alpha_1}{\lambda_1} \right| \cdot \|\vec{v}_1\| + \left| \frac{\alpha_2}{\lambda_2} \right| \cdot \|\vec{v}_2\| \\ \Delta &\leq \left| \frac{\alpha_1}{\lambda_1} \right| + \left| \frac{\alpha_2}{\lambda_2} \right| \end{aligned}$$

Plugging in the values:

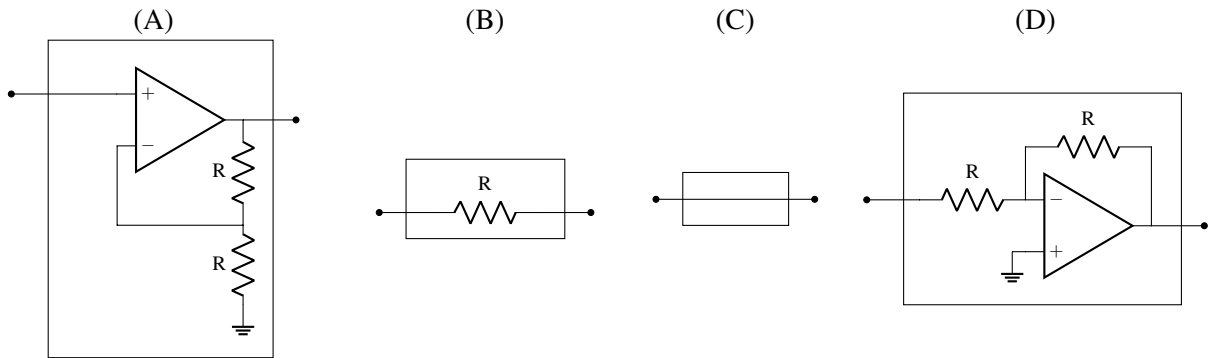
$$\begin{aligned} \Delta &\leq \left| \frac{4}{2} \right| + \left| \frac{-1}{1} \right| \\ \Delta &\leq 2 + 1 \\ \Delta &\leq 3 \end{aligned}$$

**14. Op Amp Blocks (12 points)**

For each subpart below you will be given a formula for  $V_{out} = \alpha V_1 + \beta V_2$ , where  $\alpha$  and  $\beta$  will be real numbers, and  $V_{out}, V_1, V_2$  are denoted in the Outline Schematic. Place the given circuit blocks in the outline schematic to produce the desired  $V_{out}$ .

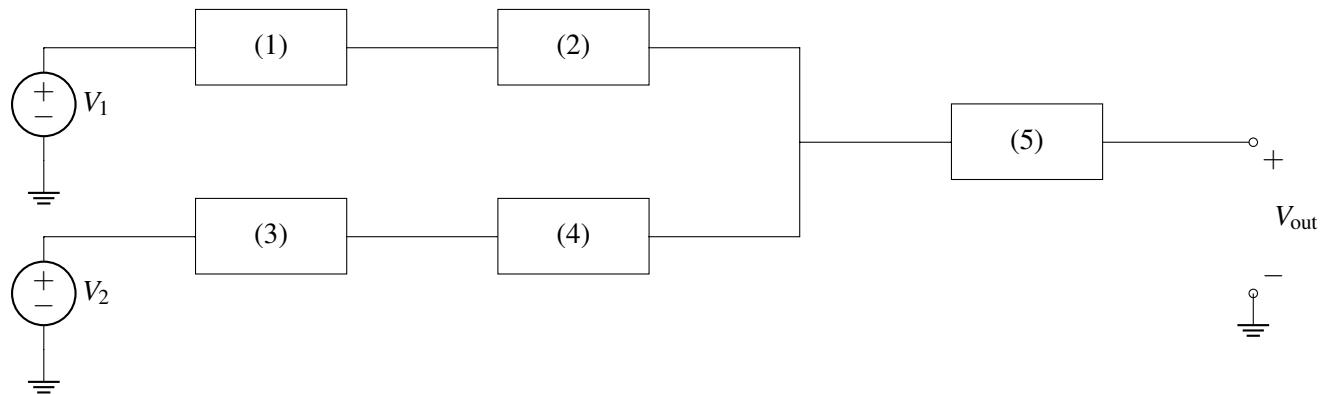
**Note:** There is more than one solutions to each subpart; you only need to find one of them per subpart.

**Circuit Blocks:**

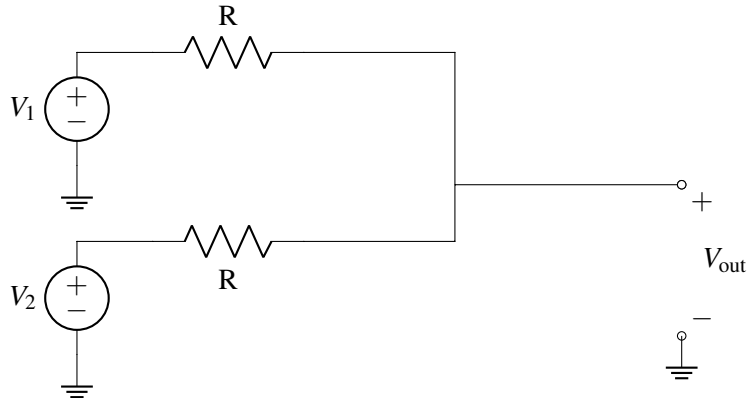


Important Design Constraints: All resistors have the same value,  $R$ . You can use the resistor and wire blocks more than once for each operation, but each op amp configuration can be used at most once for each operation.

**Outline Schematic:**



For example:  $V_{out} = \frac{1}{2}V_1 + \frac{1}{2}V_2$

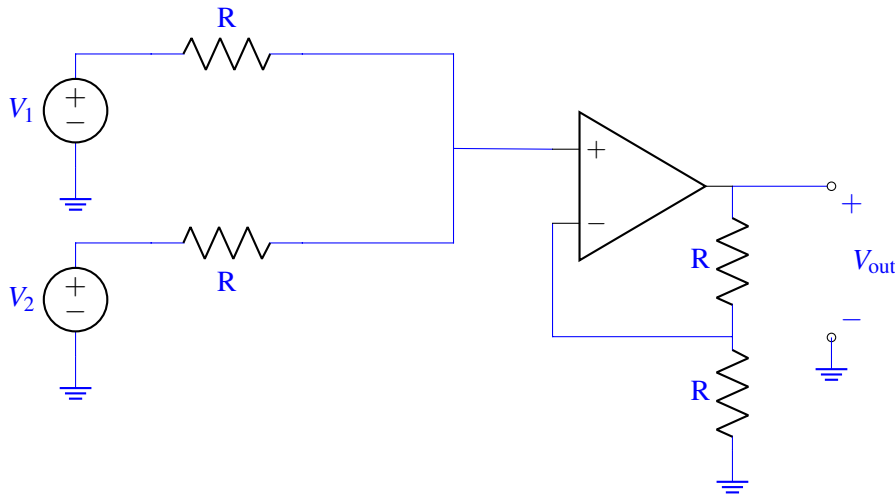


In this case, two correct answers would be BCBC and CBCB (there are more, but we only want you to find one).

**Fill in the outline schematic** for the following 3 expressions:

(a) (4 points)  $V_{out} = V_1 + V_2$

**Solution:** Looking at the summer circuit from the example above we see that we need to amplify its output by 2. Using a non-inverting amplifier topology with a gain of two we reach the solution shown below (**BCBCA**):



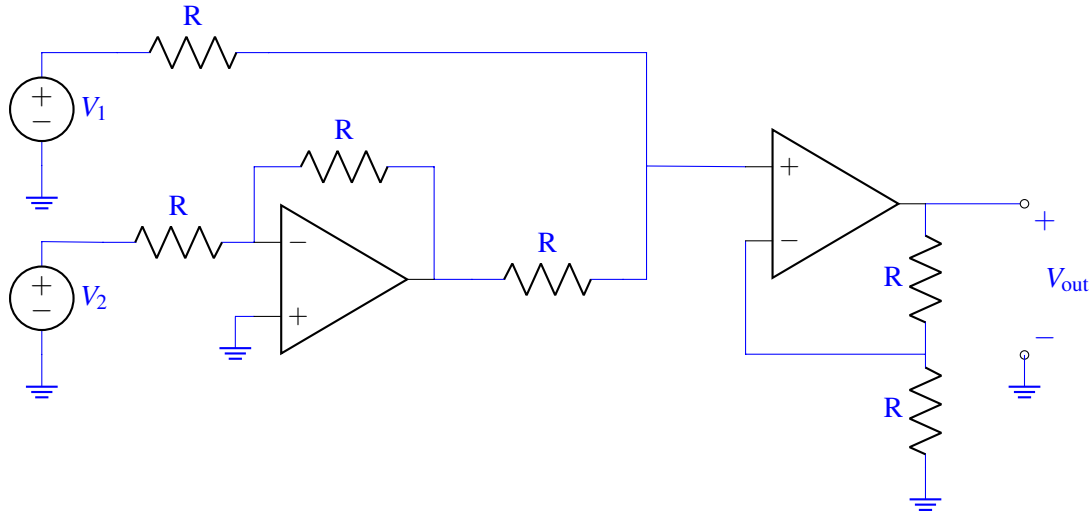
Note that the non-inverting amp does not load the summer before it and acts as an ideal driver (zero output resistance due to the VCVS at its output). Other correct (and equivalent) solutions for this topology are: **CBCBA**, **CBBCA**, **BCCBA**.



(b) (4 points)  $V_{out} = V_1 - V_2$

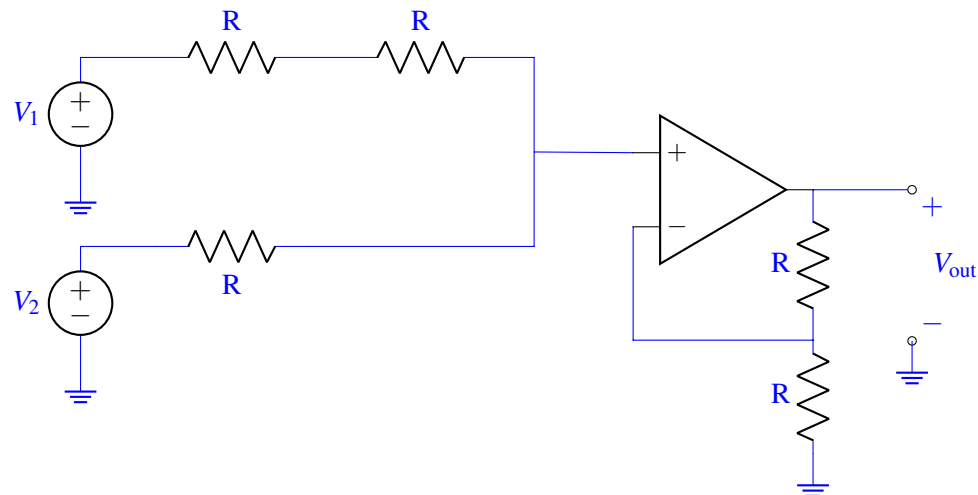
**Solution:** Notice that in this case  $V_2$  needs to be inverted and then summed to  $V_1$ . Hence, on the branch of  $V_2$  we will use an inverting amplifier topology, with a voltage gain of  $-1$ . Having created  $-V_2$  we will proceed using the same summer circuit as before to reach  $\frac{1}{2}(V_1 - V_2)$ , and then feed that into the non-inverting amplifier topology of gain 2 that has been given. This yields the two following equivalent solutions: **CBDDBA** and **BCDDBA**.

Note that, even though it is possible to create this signal output with other configurations, we cannot do so while still abiding by the given outline schematic.



(c) (4 points)  $V_{out} = \frac{2}{3}V_1 + \frac{4}{3}V_2$

**Solution:** The first thing to notice here, is that the coefficients of  $V_1$  and  $V_2$  add up to 2. Remembering that the resistive summer circuit can be used to create weights that sum up to 1, we will use that summer (we also have no restriction on the number of resistors we can use), followed by the non-inverting amplifier with a gain of 2 that we have at our disposal. Using the summer circuit to create  $\frac{1}{3}V_1 + \frac{2}{3}V_2$  at the input of the non-inverting amp we get the solution below (**BBBBA** or equivalently **BBCBA**).



**15. Orthogonality (26 points)**

(a) (4 points) Let  $\mathbf{W} = \vec{x}\vec{y}^T$  where  $\vec{x} \in \mathbb{R}^n$  and  $\vec{y} \in \mathbb{R}^m$  are orthogonal to one another.

i. What is the dimension of  $C(\mathbf{W})$ , the columnspace of  $\mathbf{W}$ ?

- (A) 0                                      (C)  $n$                                       (E)  $m$                                       (G) Not enough information  
 (B) 1                                      (D)  $n - 1$                                       (F)  $m - 1$

ii. What is the dimension of  $N(\mathbf{W})$ , the nullspace of  $\mathbf{W}$ ?

- (A) 0                                      (C)  $n$                                       (E)  $m$                                       (G) Not enough information  
 (B) 1                                      (D)  $n - 1$                                       (F)  $m - 1$

iii. Is  $\mathbf{W}$  invertible?

- (A) Yes                                      (B) No

iv. The *trace* of a square matrix is the sum of the entries along the diagonal. (For example,  $\mathbf{S} =$

$$\begin{bmatrix} s_{11} & s_{12} & \dots \\ s_{21} & s_{22} & \dots \\ \dots & \dots & s_{mm} \end{bmatrix} \text{ has a trace} = s_{11} + s_{22} + \dots + s_{mm}.) \text{ If } m = n, \text{ what is the trace of } \mathbf{W}?$$

- (A)  $\|\vec{x}\|$                                       (C)  $\|\vec{x}\| \cdot \|\vec{y}\|$                                       (E) Not enough information  
 (B)  $\|\vec{y}\|$                                       (D) 0

**Solution:** Let  $\vec{x} = [x_1 \ x_2 \ \dots \ x_n]^T$  and  $\vec{y} = [y_1 \ y_2 \ \dots \ y_m]^T$ .

$$\mathbf{W} = \vec{x}\vec{y}^T = \begin{bmatrix} x_1y_1 & x_1y_2 & \dots & x_1y_m \\ x_2y_1 & x_2y_2 & \dots & x_2y_m \\ \vdots & & \ddots & \vdots \\ x_ny_1 & x_ny_2 & \dots & x_ny_m \end{bmatrix} = \begin{bmatrix} | & | & & | \\ y_1\vec{x} & y_2\vec{x} & \dots & y_m\vec{x} \\ | & | & & | \end{bmatrix}$$

**Part (i)** By inspection, we see that all of the columns are scalar multiples for  $\vec{x}$  so the span of the columns is just the span of  $\vec{x}$ , which has dimension of 1. The columnspace of  $\mathbf{W}$  is the span of the columns of the  $\mathbf{W}$ , so the dimension of the columnspace is 1. This is also the rank of  $\mathbf{W}$ .

**Part (ii)** From the rank-nullity theorem, we know that the the number of columns in  $\mathbf{W}$  is equal to the sum of the dimensions to the columnspace and nullspace.  $\mathbf{W}$  has  $m$  columns and rank 1, so the dimension of the nullspace is  $m - 1$ .

**Part (iii)** Only matrices with a trivial nullspace are invertible. Since  $\mathbf{W}$  has a non-trivial nullspace (with dimension  $m - 1$  from part (ii)) then it is not invertible.

**Part (iv)** If  $m = n$ , the trace of  $\mathbf{W}$  is  $x_1y_1 + x_2y_2 + \dots + x_ny_n = \vec{x}^T\vec{y}$ . Since we know that  $\vec{x}$  and  $\vec{y}$  are orthogonal,  $\vec{x}^T\vec{y} = 0$ , so the trace is 0.

(b) (4 points) Recall that an orthonormal matrix is a matrix that satisfies the following two properties:

- All columns have norm of 1

- All columns are orthogonal to one another

Let  $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ,  $\vec{u} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ .

Which of the following matrices are orthonormal? Choose all that apply.

(1)  $\begin{bmatrix} | & | \\ \text{proj}_{\vec{e}_1}(\vec{v}) & \text{proj}_{\vec{e}_2}(\vec{v}) \\ | & | \end{bmatrix}$       (2)  $\begin{bmatrix} | & | \\ \text{proj}_{\vec{e}_1}(\vec{u}) & \text{proj}_{\vec{e}_2}(\vec{u}) \\ | & | \end{bmatrix}$       (3)  $\begin{bmatrix} | & | \\ \frac{1}{5}\vec{u} & \frac{1}{\sqrt{2}}\vec{v} \\ | & | \end{bmatrix}$

(4)  $\begin{bmatrix} | & | \\ \text{proj}_{\vec{v}}(\vec{e}_1) & \text{proj}_{\vec{v}}(\vec{e}_2) \\ | & | \end{bmatrix}$       (5)  $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$       (6)  $\begin{bmatrix} | & | \\ \frac{1}{\sqrt{2}}\vec{v} & \vec{e}_2 \\ | & | \end{bmatrix}$

**Solution:**

We check each of the matrices to see if they are orthonormal:

(1)  $\begin{bmatrix} | & | \\ \text{proj}_{\vec{e}_1}(\vec{v}) & \text{proj}_{\vec{e}_2}(\vec{v}) \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \frac{\langle \vec{e}_1, \vec{v} \rangle}{\langle \vec{e}_1, \vec{e}_1 \rangle} \vec{e}_1 & \frac{\langle \vec{e}_2, \vec{v} \rangle}{\langle \vec{e}_2, \vec{e}_2 \rangle} \vec{e}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{e}_1 & -\vec{e}_2 \\ | & | \end{bmatrix}$   
 $\langle \vec{e}_1, -\vec{e}_2 \rangle = 0$  and  $\|\vec{e}_1\| = \|\vec{e}_2\| = 1$  so the matrix is **orthonormal**.

(2)  $\begin{bmatrix} | & | \\ \text{proj}_{\vec{e}_1}(\vec{u}) & \text{proj}_{\vec{e}_2}(\vec{u}) \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \frac{\langle \vec{e}_1, \vec{u} \rangle}{\langle \vec{e}_1, \vec{e}_1 \rangle} \vec{e}_1 & \frac{\langle \vec{e}_2, \vec{u} \rangle}{\langle \vec{e}_2, \vec{e}_2 \rangle} \vec{e}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ -3\vec{e}_1 & 4\vec{e}_2 \\ | & | \end{bmatrix}$   
 $\langle -3\vec{e}_1, 4\vec{e}_2 \rangle = 0$  but  $\|-3\vec{e}_1\| = 3$  and  $\|4\vec{e}_2\| = 4$  so the columns are orthogonal but don't have norm 1, so the matrix is **not orthonormal**.

(3)  $\langle \frac{1}{5}\vec{u}, \frac{1}{\sqrt{2}}\vec{v} \rangle = \frac{1}{5\sqrt{2}} \langle \vec{u}, \vec{v} \rangle = \frac{1}{5\sqrt{2}}(-3+4) = \frac{1}{5\sqrt{2}}$   
 $\|\frac{1}{5}\vec{u}\| = 1$  and  $\|\frac{1}{\sqrt{2}}\vec{v}\| = 1$   
 Therefore the columns have norm 1 but are not orthogonal, so the matrix is **not orthonormal**.

(4)  $\begin{bmatrix} | & | \\ \text{proj}_{\vec{v}}(\vec{e}_1) & \text{proj}_{\vec{v}}(\vec{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \frac{\langle \vec{e}_1, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v} & \frac{\langle \vec{e}_2, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v} \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \frac{1}{2}\vec{v} & \frac{1}{2}\vec{v} \\ | & | \end{bmatrix}$   
 The columns are the same so they cannot be orthogonal. In addition the norms of the columns are not 1. Therefore the matrix is **not orthonormal**.

(5) Let  $\vec{a}_1 = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$  and  $\vec{a}_2 = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$ .  
 $\langle \vec{a}_1, \vec{a}_2 \rangle = -\cos \theta \sin \theta + \sin \theta \cos \theta = 0$   
 $\|\vec{a}_1\| = \sqrt{\cos^2 \theta + \sin^2 \theta} = \sqrt{1} = 1$   
 $\|\vec{a}_2\| = \sqrt{\cos^2 \theta + \sin^2 \theta} = \sqrt{1} = 1$   
 The columns are orthogonal and have norm 1 so the matrix is **orthonormal**. (This is the rotation matrix!)

(6)  $\langle \frac{1}{\sqrt{2}}\vec{v}, \vec{e}_2 \rangle = \frac{1}{\sqrt{2}} \langle \vec{v}, \vec{e}_2 \rangle = \frac{1}{\sqrt{2}}$   
 $\|\frac{1}{\sqrt{2}}\vec{v}\| = 1$  and  $\|\vec{e}_2\| = 1$

Therefore the columns have norm 1 but are not orthogonal, so the matrix is **not orthonormal**.

(c) (6 points) Prove that multiplying by an orthonormal matrix does not change the inner product:

$$\langle \mathbf{M}\vec{u}, \mathbf{M}\vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle,$$

where  $\mathbf{M} \in \mathbb{R}^{n \times n}$  is an orthonormal matrix with columns  $\vec{m}_1 \dots \vec{m}_n$ . Finish the proof by filling in the numbered boxes with one of the corresponding multiple-choice options. Some boxes are intentionally left blank.

$$\begin{aligned} \langle \mathbf{M}\vec{u}, \mathbf{M}\vec{v} \rangle &= \boxed{\phantom{000}}(\mathbf{M}\vec{v}) \\ &= \boxed{(1)}\mathbf{M}\vec{v} \end{aligned} \quad (*)$$

Let  $(\mathbf{M}^T\mathbf{M})_{ij}$  be the element in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column of  $\mathbf{M}^T\mathbf{M}$ .  $(\mathbf{M}^T\mathbf{M})_{ij}$  is equal to the inner product of the  $i^{\text{th}}$  row of  $\mathbf{M}^T$  and the  $j^{\text{th}}$  column of  $\boxed{(2)}$ . By the definition of  $\boxed{(3)}$ , the  $i^{\text{th}}$  row of  $\mathbf{M}^T$  is equal to  $\boxed{(4)}$ . Therefore:

$$\begin{aligned} (\mathbf{M}^T\mathbf{M})_{ij} &= \boxed{(5)} \\ \text{If } i = j, \text{ then } (\mathbf{M}^T\mathbf{M})_{ij} &= \boxed{(6)} \text{ because } \boxed{(7)} \\ \text{If } i \neq j, \text{ then } (\mathbf{M}^T\mathbf{M})_{ij} &= \boxed{(8)} \text{ because } \boxed{(9)} \end{aligned}$$

Therefore,  $\mathbf{M}^T\mathbf{M} = \boxed{\phantom{000}}$ . Going back to Eq. (\*) and plugging this in yields:

$$\begin{aligned} \langle \mathbf{M}\vec{u}, \mathbf{M}\vec{v} \rangle &= \vec{u}^T I \vec{v} \\ &= \vec{u}^T \vec{v} \\ &= \langle \vec{u}, \vec{v} \rangle \end{aligned}$$

- |   |  |   |
|---|--|---|
| (1) (A) $\mathbf{M}^T \vec{u}^T$                | (2) (A) $\vec{u}^T \mathbf{M}^T$                               | (3) (A) inner product                               |
| (B) $\vec{u}^T \mathbf{M}^T$                    | (B) $\mathbf{M}\vec{v}$  | (B) matrix multiplication                           |
| (C) $\mathbf{M}^T$                              | (C) $\mathbf{M}^T$   | (C) transpose                                       |
| (D) $\vec{u}\mathbf{M}^T$                       | (D) $\mathbf{M}$   | (D) orthogonality                                   |
| (4) (A) $\vec{m}_i$                             | (5) (A) $\langle \mathbf{M}\vec{u}, \mathbf{M}\vec{v} \rangle$ | (6) (A) 0   |
| (B) $\vec{v}$                                   | (B) $\langle \vec{m}_i, \vec{m}_j \rangle$                     | (B) $i$   |
| (C) $\vec{u}$                                   | (C) $\langle \vec{m}_i^T \vec{u}, \vec{m}_j^T \vec{v} \rangle$ | (C) $j$   |
| (D) $\vec{m}_j$                                 | (D) $\langle \vec{u}, \vec{v} \rangle$                         | (D) 1   |
| (7) (A) $\mathbf{M}\mathbf{M}^T = I$            | (8) (A) $i$  | (9) (A) $\ \vec{m}_k\ ^2 = 1$ for $1 \leq k \leq n$ |
| (B) $\ \vec{m}_k\ ^2 = 1$ for $1 \leq k \leq n$ | (B) 0  | (B) $\mathbf{M}\mathbf{M}^T = I$                    |
| (C) the columns of $\mathbf{M}$ are orthogonal  | (C) 1  | (C) $\mathbf{M}$ is a square matrix                 |
| (D) $\mathbf{M}$ is a square matrix             | (D) $j$  | (D) the columns of $\mathbf{M}$ are orthogonal      |

**Solution:**

$$\begin{aligned} \langle \mathbf{M}\vec{u}, \mathbf{M}\vec{v} \rangle &= (\mathbf{M}\vec{u})^T (\mathbf{M}\vec{v}) \\ &= \vec{u}^T \mathbf{M}^T \mathbf{M} \vec{v} \end{aligned} \quad (*)$$

Let  $(\mathbf{M}^T \mathbf{M})_{ij}$  be the element in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column of  $\mathbf{M}^T \mathbf{M}$ .  $(\mathbf{M}^T \mathbf{M})_{ij}$  is equal to the inner product of the  $i^{\text{th}}$  row of  $\mathbf{M}^T$  and the  $j^{\text{th}}$  column of  $\mathbf{M}$ . By the definition of **transpose**, the  $i^{\text{th}}$  row of  $\mathbf{M}^T$  is equal to  $\vec{m}_i$ . Therefore:

$$\begin{aligned} (\mathbf{M}^T \mathbf{M})_{ij} &= \langle \vec{m}_i, \vec{m}_j \rangle \\ \text{If } i = j, \text{ then } (\mathbf{M}^T \mathbf{M})_{ij} &= 1 \text{ because } \|\vec{m}_k\|^2 = 1 \text{ for } 1 \leq k \leq n \\ \text{If } i \neq j, \text{ then } (\mathbf{M}^T \mathbf{M})_{ij} &= 0 \text{ because the columns of } \mathbf{M} \text{ are orthogonal} \end{aligned}$$

Therefore,  $\mathbf{M}^T \mathbf{M} = \mathbf{I}$ . Going back to Eq. (\*) and plugging this in yields:

$$\begin{aligned} \langle \mathbf{M}\vec{u}, \mathbf{M}\vec{v} \rangle &= \vec{u}^T \mathbf{I} \vec{v} \\ &= \vec{u}^T \vec{v} \\ &= \langle \vec{u}, \vec{v} \rangle \end{aligned}$$

- (d) (4 points) Let  $\mathbf{V}$  and  $\mathbf{U}$  be orthonormal matrices in  $\mathbb{R}^{n \times n}$  with columns  $\vec{v}_1 \dots \vec{v}_n$  and  $\vec{u}_1 \dots \vec{u}_n$ , respectively. Let  $\mathbf{D}$  be a diagonal matrix in  $\mathbb{R}^{n \times n}$  with non-zero scalars  $d_1 \dots d_n$  along the diagonal and zeros

everywhere else, i.e.  $\mathbf{D} = \begin{bmatrix} d_1 & 0 & \dots \\ 0 & d_2 & \dots \\ \dots & \dots & \dots \end{bmatrix}$ . If

$$\mathbf{W} = \mathbf{U}\mathbf{D}\mathbf{V}^T,$$

which of the following must be true? Select all that apply. Assume  $1 \leq i \leq n$ .

- (1)  $\mathbf{D} = \mathbf{V}\mathbf{W}\mathbf{U}^T$
- (2)  $\vec{u}_i^T \mathbf{W} = d_i \vec{v}_i^T$
- (3)  $\mathbf{W}\vec{u}_i = d_i \vec{v}_i$
- (4)  $\mathbf{W}\vec{v}_i = d_i \vec{u}_i$
- (5)  $\mathbf{W}^{-1} = \mathbf{V}\mathbf{D}^{-1}\mathbf{U}^T$

**Solution:** We showed in discussion that  $\mathbf{A}^{-1} = \mathbf{A}^T$  if  $\mathbf{A}$  is an orthonormal matrix. We'll apply this property in this problem.

- (1) We start with the given statement and find an expression for  $\mathbf{D}$ :

$$\begin{aligned} \mathbf{W} &= \mathbf{U}\mathbf{D}\mathbf{V}^T \\ \mathbf{W}\mathbf{V} &= \mathbf{U}\mathbf{D}\mathbf{V}^T \mathbf{V} \\ \mathbf{W}\mathbf{V} &= \mathbf{U}\mathbf{D} \quad \text{because } \mathbf{V}^T = \mathbf{V}^{-1} \\ \mathbf{U}^T \mathbf{W}\mathbf{V} &= \mathbf{U}^T \mathbf{U}\mathbf{D} \\ \mathbf{U}^T \mathbf{W}\mathbf{V} &= \mathbf{D} \quad \text{because } \mathbf{U}^T = \mathbf{U}^{-1} \end{aligned}$$

This is not the same expression as given in choice (1) so **choice (1) is false**.

(2) We start by finding an expression for  $\vec{u}_i^T \mathbf{W}$ :

$$\begin{aligned}
 \vec{u}_i^T \mathbf{W} &= \vec{u}_i^T \mathbf{U} \mathbf{D} \mathbf{V}^T \\
 &= [- \vec{u}_i^T \quad -] \begin{bmatrix} | & | & & | & & | \\ \vec{u}_1 & \vec{u}_2 & \dots & \vec{u}_i & \dots & \vec{u}_n \\ | & | & & | & & | \end{bmatrix} \begin{bmatrix} d_1 & 0 & \dots & \dots & \dots & 0 \\ 0 & d_2 & \dots & \dots & \dots & 0 \\ \vdots & & \ddots & & & 0 \\ \vdots & & & d_i & & 0 \\ \vdots & & & & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & d_n \end{bmatrix} \begin{bmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ & \vdots & \\ - & \vec{v}_i^T & - \\ & \vdots & \\ - & \vec{v}_n^T & - \end{bmatrix} \\
 &= [0 \quad 0 \quad \dots \quad 1 \quad \dots \quad 0] \begin{bmatrix} d_1 & 0 & \dots & \dots & \dots & 0 \\ 0 & d_2 & \dots & \dots & \dots & 0 \\ \vdots & & \ddots & & & 0 \\ \vdots & & & d_i & & 0 \\ \vdots & & & & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & d_n \end{bmatrix} \begin{bmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ & \vdots & \\ - & \vec{v}_i^T & - \\ & \vdots & \\ - & \vec{v}_n^T & - \end{bmatrix} \\
 &= [0 \quad 0 \quad \dots \quad d_i \quad \dots \quad 0] \begin{bmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ & \vdots & \\ - & \vec{v}_i^T & - \\ & \vdots & \\ - & \vec{v}_n^T & - \end{bmatrix} \\
 &= d_i \vec{v}_i^T
 \end{aligned}$$

Therefore, **choice (2) is true.**

(3) We attempt to calculate  $\mathbf{W}\vec{u}_i$ :

$$\begin{aligned}
 \mathbf{W}\vec{u}_i &= \mathbf{U} \mathbf{D} \mathbf{V}^T \vec{u}_i \\
 &= \mathbf{U} \begin{bmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ & \vdots & \\ - & \vec{v}_i^T & - \\ & \vdots & \\ - & \vec{v}_n^T & - \end{bmatrix} \vec{u}_i = \mathbf{U} \begin{bmatrix} - & \vec{v}_1^T \vec{u}_i & - \\ - & \vec{v}_2^T \vec{u}_i & - \\ & \vdots & \\ - & \vec{v}_i^T \vec{u}_i & - \\ & \vdots & \\ - & \vec{v}_n^T \vec{u}_i & - \end{bmatrix}
 \end{aligned}$$

We don't know that the columns of  $\mathbf{U}$  and  $\mathbf{V}$  are orthogonal to each other, so we cannot simplify this expression further. Therefore, **choice (3) is false.**

(4) We now calculate  $\mathbf{W}\vec{v}_i$ :

$$\begin{aligned}
 \mathbf{W}\vec{u}_i &= \mathbf{U}\mathbf{D}\mathbf{V}^T\vec{v}_i \\
 &= \mathbf{U}\mathbf{D} \begin{bmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ & \vdots & \\ - & \vec{v}_i^T & - \\ & \vdots & \\ - & \vec{v}_n^T & - \end{bmatrix} \vec{v}_i = \mathbf{U}\mathbf{D} \begin{bmatrix} - & \vec{v}_1^T\vec{v}_i & - \\ - & \vec{v}_2^T\vec{v}_i & - \\ & \vdots & \\ - & \vec{v}_i^T\vec{v}_i & - \\ & \vdots & \\ - & \vec{v}_n^T\vec{v}_i & - \end{bmatrix} \\
 &= \mathbf{U} \begin{bmatrix} d_1 & 0 & \dots & \dots & \dots & 0 \\ 0 & d_2 & \dots & \dots & \dots & 0 \\ \vdots & & \ddots & & & 0 \\ \vdots & & & d_i & & 0 \\ \vdots & & & & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & d_n \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} | & | & \dots & | & \dots & | \\ \vec{u}_1 & \vec{u}_2 & \dots & \vec{u}_i & \dots & \vec{u}_n \\ | & | & & | & & | \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ d_i \\ \vdots \\ 0 \end{bmatrix} \\
 &= d_i\vec{u}
 \end{aligned}$$

Therefore, **choice (4) is true.**

(5) We check if  $\mathbf{W}^{-1} = \mathbf{V}\mathbf{D}^{-1}\mathbf{U}^T$  is true by multiplying the expression for  $\mathbf{W}^{-1}$  with  $\mathbf{W}$  and seeing if we get the identity matrix:

$$\begin{aligned}
 (\mathbf{V}\mathbf{D}^{-1}\mathbf{U}^T)\mathbf{W} &= (\mathbf{V}\mathbf{D}^{-1}\mathbf{U}^T)(\mathbf{U}\mathbf{D}\mathbf{V}^T) \\
 &= \mathbf{V}\mathbf{D}^{-1}(\mathbf{U}^T\mathbf{U})\mathbf{D}\mathbf{V}^T \\
 &= \mathbf{V}(\mathbf{D}^{-1}\mathbf{D})\mathbf{V}^T \\
 &= \mathbf{V}\mathbf{V}^T \\
 &= \mathbf{I}
 \end{aligned}$$

Here, we used the property that  $\mathbf{U}^{-1} = \mathbf{U}^T$  and  $\mathbf{V}^{-1} = \mathbf{V}^T$ . We find that **choice (5) is true**

- (e) (8 points) Let  $\mathbf{W}$  be a square, symmetric  $n \times n$  matrix ( $\mathbf{W} = \mathbf{W}^T$ ). Prove that the columnspace of  $\mathbf{W}$  is orthogonal to the nullspace of  $\mathbf{W}$ . In other words, prove that any vector,  $\vec{v}$ , in the columnspace of  $\mathbf{W}$  is orthogonal to any vector,  $\vec{u}$ , in the nullspace of  $\mathbf{W}$ . Justify every step of your proof.

**Solution:** Let  $\vec{v}$  be an arbitrary vector in the columnspace of  $\mathbf{W}$  and let  $\vec{u}$  be an arbitrary vector in the nullspace of  $\mathbf{W}$ . By the definition of nullspace

$$\mathbf{W}\vec{u} = \vec{0}, \quad (7)$$

and by the definition of columnspace, there exists some vector  $\vec{b} \in \mathbb{R}^n$  such that

$$\mathbf{W}\vec{b} = \vec{v}. \quad (8)$$

Since  $\mathbf{W} = \mathbf{W}^T$  we can also write the following statements which follow from Eq. 7 and Eq. 8.

$$\mathbf{W}^T\vec{u} = \vec{0}, \quad (9)$$

$$\mathbf{W}^T\vec{b} = \vec{v}. \quad (10)$$

We start with Eq. 10 and take the transpose of both sides:

$$\vec{b}^T \mathbf{W} = \vec{v}^T$$

Then we right-multiply both sides by  $\vec{u}$ :

$$\vec{b}^T \mathbf{W} \vec{u} = \vec{v}^T \vec{u}$$

We plug in the expression for  $\mathbf{W}\vec{u}$  from Eq. 7:

$$\vec{b}^T \vec{0} = \vec{v}^T \vec{u}$$

$$0 = \vec{v}^T \vec{u}$$

We've shown that  $\vec{v}^T \vec{u} = \langle \vec{v}, \vec{u} \rangle = 0$  so the vectors  $\vec{v}$  and  $\vec{u}$  are orthogonal. Since there were generic vectors in the columnspace and nullspace, respectively, we've shown that all vectors in the columnspace are orthogonal to all vectors in the nullspace.



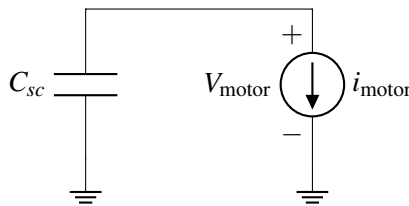
**16. Feedback Formula (33 points)**

Formula One (F1) is widely considered as the pinnacle of motorsport. Recently its competitiveness has come into question, since the same franchises have dominated the sport over the past decades.

- (a) (3 points) In order to increase overtaking and thus competition, F1 is introducing electric motors that provide an instant boost to drivers if they are within a short distance of the driver ahead of them.

This instant boost will be coming from none other than a super-capacitor. During boost operation, the motor requires a constant current,  $i_{\text{motor}} = 4\text{A}$ . It also has a nominal voltage of  $V_{\text{motor, nom}} = 10\text{V}$  but can operate with voltages as low as  $V_{\text{motor, min}} = 8\text{V}$ . When the voltage drops below  $V_{\text{motor, min}}$ , all extra acceleration is lost and the car resumes at its previous speed.

In order for the car to have a good chance to successfully overtake, the boost needs to be maintained for at least  $t_{\text{overtake}} = 5\text{s}$ . What is the minimum value of the super-capacitor  $C_{sc}$  that allows the boost to be maintained for at least  $t_{\text{overtake}}$ ?



**Solution:** The super-capacitor voltage will be constantly dropping starting from  $V_{\text{motor, nom}} = 10\text{V}$ , down to  $V_{\text{motor, min}} = 8\text{V}$ , at which point the acceleration will be cut off. Using that  $I_{sc} = C \frac{dV_{sc}}{dt} = i_{\text{motor}}$ , we can solve for the minimum value of the super-capacitor required to maintain acceleration for at least  $t_{\text{overtake}}$ :

$$\begin{aligned}
 I_{sc} &= i_{\text{motor}} = C_{sc} \frac{dV_{\text{motor}}}{dt} \\
 \Rightarrow C_{sc} &= \frac{i_{\text{motor}}}{V_{\text{nom}} - V_{\text{min}}} t_{\text{overtake}} \\
 \Rightarrow C_{sc} &= \frac{4\text{A}}{10\text{V} - 8\text{V}} 5\text{sec} \\
 \Rightarrow C_{sc} &= 10\text{F}
 \end{aligned}$$

(b) (2 points) Given that super-capacitors can be discharged pretty quickly you come up with a backup acceleration plan: you add one more motor to your system. Unlike the previous motor that was modeled as a constant current source, this motor can be thought of as a load resistor, whose rotational speed increases as the voltage across it increases.

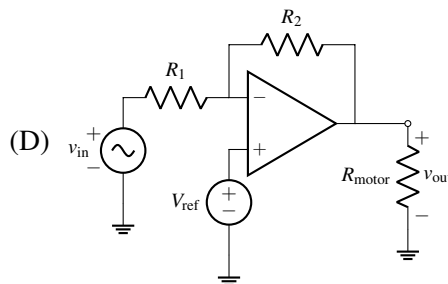
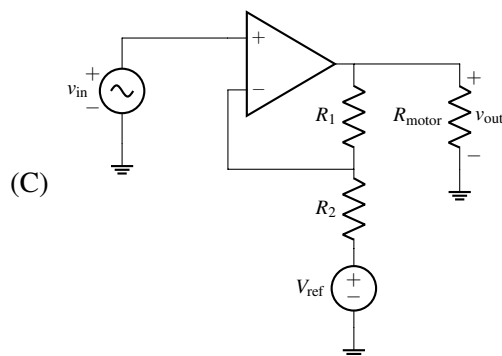
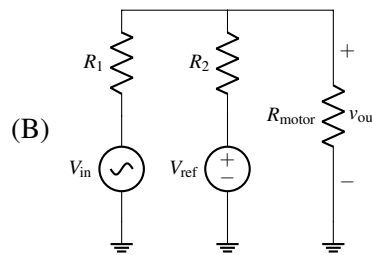
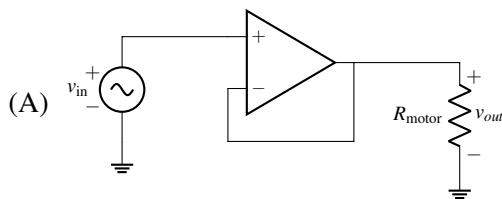
You want to design a system that **increases the voltage provided to the motor as the distance between a given driver and the driver leading them decreases**, making it maximum when the distance between them goes to zero ( $V_{R_{\text{motor, max}}} = 10\text{V}$ ). At the same time, you don't want to be spending extra energy if you are far away from the leading driver, so the acceleration motor should shut down (i.e.  $V_{R_{\text{motor, min}}} = 0\text{V}$ ) when the distance is larger than 15m.

You are given the following components:

- i. one Op Amp (assume that it comes with  $V_{\text{DD}}$  and  $V_{\text{SS}}$  that are sufficiently large so you don't need to worry about those)
- ii. one motor that operates in the 0-10V range - modeled as  $R_{\text{motor}}$
- iii. one LIDAR sensor that returns a voltage in the 0-10V range directly proportional to distance. Specifically, when the distance is 0m the output of the sensor is 0V, and when the distance is larger than 15m the output of the sensor is "maxed-out" at 10V. The voltage grows linearly in all intermediate distances. This sensor is modeled as a variable voltage source  $v_{\text{in}}$ .
- iv. one constant voltage source ( $V_{\text{ref}}$ ) set at a voltage of your choice
- v. 2 resistors ( $R_1$  and  $R_2$ ) of your choice

You may or may not need all of them.

Which of the following topologies should you pick to build this system?



**Solution:** As distance ( $v_{\text{in}}$ ) decreases, the motor voltage ( $v_{\text{out}}$ ) needs to be increasing. This suggests an inversely proportional input-output relationship. An inverting amplifier must be used, with a reference voltage at the non-inverting terminal. This reference voltage will set the maximum output voltage (when the sensor input is zero). This way, if the sensor reading goes to zero, the output will

be maximized. Conversely, if the sensor reading starts increasing, the output will be decreasing since the input appears inverted at the output and will be subtracted from the maximum output voltage value that is set by  $V_{\text{ref}}$ . The answer is (D).

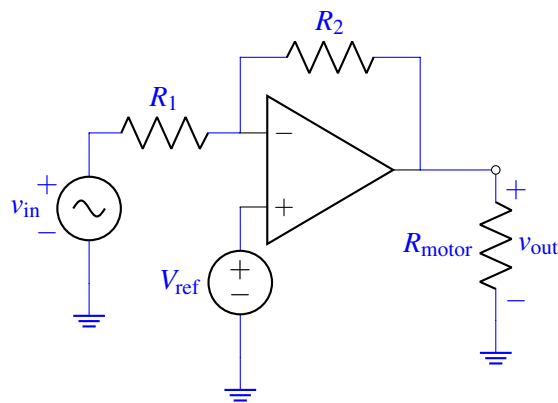
A voltage summer cannot be used as it simply adds a constant voltage  $V_{\text{ref}}$  to  $v_{\text{in}}$ .

A unity buffer cannot be used as it sets  $v_{\text{out}} = v_{\text{in}}$ .

A non-inverting amplifier increases  $v_{\text{out}}$  as  $v_{\text{in}}$  increases. It is used for directly proportional relationships.

- (c) (8 points) **Redraw** your choice from part (b) and **explain** how your design satisfies the specifications in part (b). **Find numerical values** for  $R_1$ ,  $R_2$ , and  $V_{\text{ref}}$  for the topology you picked in the previous part (or for the subset of these components in the topology you picked).

**Solution:**



For an inverting amplifier:

$$v_{\text{out}} = v_{\text{in}} \left( -\frac{R_2}{R_1} \right) + V_{\text{ref}} \left( \frac{R_2}{R_1} + 1 \right)$$

When the cars are very close, we want the motor to spin the fastest, i.e., when  $v_{\text{in}} = 0$ ,  $v_{\text{out}} = 10$ .

$$\begin{aligned} \therefore v_{\text{out}} &= 0 + V_{\text{ref}} \left( \frac{R_2}{R_1} + 1 \right) \\ 10 &= V_{\text{ref}} \left( \frac{R_2}{R_1} + 1 \right) \end{aligned}$$

When the cars are far apart, we don't want the motor to spin, i.e., when  $v_{\text{in}} = 10$ ,  $v_{\text{out}} = 0$ .

$$\begin{aligned} \therefore v_{\text{out}} = 0 &= -10 \frac{R_2}{R_1} + V_{\text{ref}} \left( \frac{R_2}{R_1} + 1 \right) \\ \text{But from above we have, } V_{\text{ref}} \left( \frac{R_2}{R_1} + 1 \right) &= 10 \\ 10 \frac{R_2}{R_1} &= 10 \\ \Rightarrow \frac{R_2}{R_1} &= 1 \text{ or } R_1 = R_2 \\ \text{Substituting into } V_{\text{ref}} \left( \frac{R_2}{R_1} + 1 \right) &= 10, \\ \text{we get } V_{\text{ref}} &= \frac{10}{2} = 5V \end{aligned}$$

We can pick any resistor values that are identical for  $R_1$  and  $R_2$  (the larger we pick them, the lower the power consumption will be, but no such constraint was set in this problem).

(d) (10 points) Safety regulations in F1 impose an upper speed limit, so you need to make sure the acceleration boost is cut-off before you reach that limit. However, your car only has an **accelerometer**. Fill in the boxes in the outline schematic to **build a circuit that can measure speed, (i.e. produce an output voltage proportional to speed)**, using the following components:

- i. **one** Op Amp (assume that it comes with  $V_{DD}$  and  $V_{SS}$  that are sufficiently large so you don't need to worry about those)
- ii. an accelerometer that returns a voltage (0-5V) directly proportional to acceleration - modeled as a variable voltage source  $v_{in}$
- iii. **one** resistor  $R$  of your choice
- iv. **one** capacitor  $C$  of your choice
- v. unlimited wires

**Draw** the design you end up with after filling the boxes and **explain** how it works. **Express** the output voltage  $v_{out}$  as a function of component values,  $R$ ,  $C$ , and the input voltage  $v_{in}$ .

*Hint:* Remember, speed can be found as the integral of acceleration.

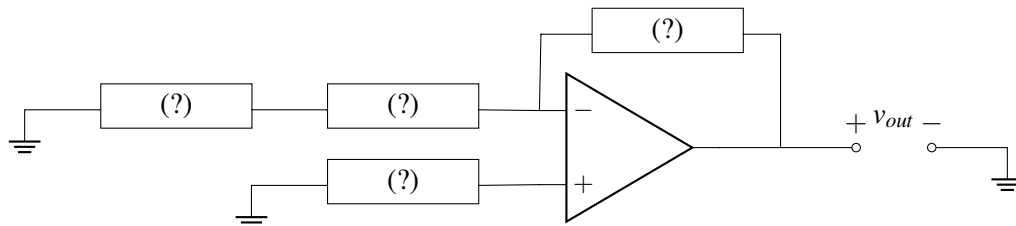
**Note:** The sign of the output does not matter.

The circuit blocks you have available are repeated below for your convenience.

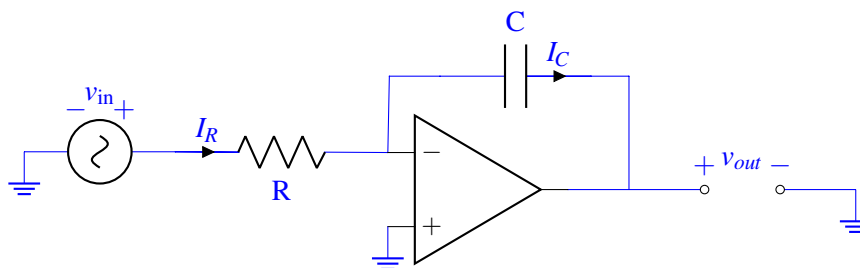
**Circuit Blocks:**



**Outline Schematic:**



**Solution:** Velocity (speed) can be written as the integral of acceleration. Thus, we require an integrator circuit.



Using GR #2, and Ohm's law we get:

$$u^- = u^+ = 0V$$

$$I_R = \frac{V_R}{R} = \frac{v_{in}}{R}$$

By KCL and GR #1 on  $u^-$ , we get:

$$I_C = I_R$$

The capacitor characteristic equation gives us:

$$I_C = C \frac{dV_C}{dt} = C \frac{d(0 - v_{out})}{dt} = C \frac{-dv_{out}}{dt}$$

Combining the previous three equations we can write:

$$\begin{aligned} \frac{v_{in}}{R} &= C \frac{-dv_{out}}{dt} \\ \Rightarrow v_{out}(t) &= -\frac{1}{RC} \int_0^t v_{in} dt + V_C(0) \end{aligned}$$

Thus, this circuit does indeed integrate the input as was required.

(e) (10 points) Alas! Some feisty wheel-to-wheel battles have broken your accelerometer. You are now stuck with only an **odometer** (a device that measures the distance you travel over time and converts it to a voltage). Use the exact same components to **build a backup speedometer**.

- i. **one** Op Amp (assume that it comes with  $V_{DD}$  and  $V_{SS}$  that are sufficiently large so you don't need to worry about those)
- ii. an odometer that returns a voltage (0-5V) directly proportional to distance traveled - modeled as a variable voltage source  $v_{in}$
- iii. **one** resistor  $R$  of your choice
- iv. **one** capacitor  $C$  of your choice
- v. unlimited wires

**Draw** the design you end up with after filling the boxes and **explain how it works**. Express the output voltage  $v_{out}$  as a function of component values,  $R$ ,  $C$ , and the input voltage  $v_{in}$ .

*Hint:* Remember, speed can be found as the derivative of position. Which component that you have seen in class can be used to take a derivative of voltage and how can you use it in this design?

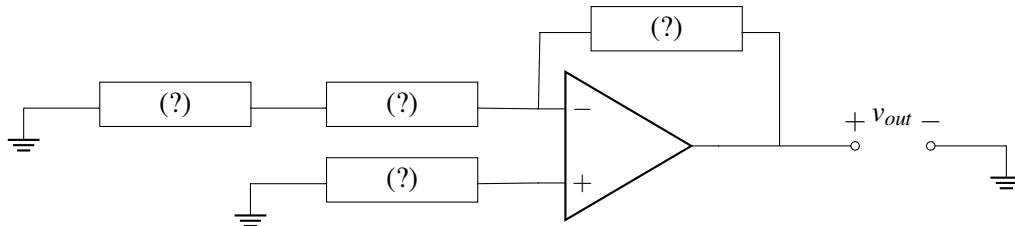
**Note:** The sign of the output, again does not matter.

The circuit blocks you have available are repeated below for your convenience.

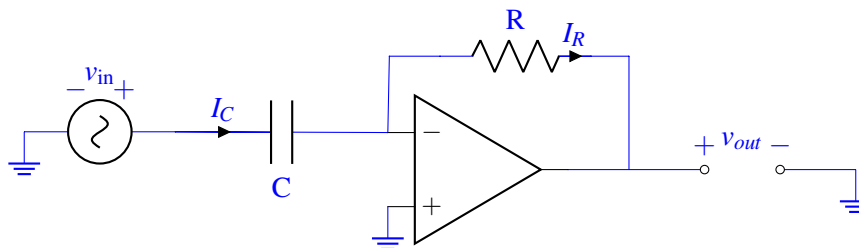
**Circuit Blocks:**



**Outline Schematic:**



**Solution:** Speed can be written as the differential of distance. Thus, we require an differentiator circuit.



GR #2 and the capacitor characteristic equation gives us:

$$u^- = u^+ = 0V$$

$$I_C = C \frac{dV_C}{dt} = C \frac{d(v_{in} - 0)}{dt} = C \frac{dv_{in}}{dt}$$

By KCL and GR #1 on  $u^-$ , we get:

$$I_C = I_R$$

Ohm's law in this case is:

$$I_R = \frac{V_R}{R} = \frac{0 - v_{out}}{R}$$

Finally, combining the previous three equations we can write:

$$\begin{aligned} \frac{-v_{out}}{R} &= C \frac{dv_{in}}{dt} \\ \Rightarrow v_{out} &= -RC \frac{dv_{in}}{dt} \end{aligned}$$

Thus, this circuit does indeed differentiate the input as was required.