15.1 Introduction: Superposition

Circuit analysis can be cumbersome to do by hand, and it can be difficult to understand the high-level behavior of complicated circuits given a schematic. We need tools we can use to both lessen the burden of analysis, and help us think about circuits and understand how they behave. Ultimately, our goal is to design interesting circuits, and in this note we will build up additional tools to help us.

15.2 Dependent Sources

At this point, we will introduce a new circuit element: dependent sources. Dependent sources act like the independent sources we’ve studied so far, but instead of producing a single voltage/current, the voltage/current is controlled by something else in the circuit. There are four different types of dependent sources, shown below:

- **Voltage-controlled voltage source**
  - \(+\) \(V_i\)
  - \(-\) \(\alpha V_i\)

- **Voltage-controlled current source**
  - \(+\) \(V_i\)
  - \(-\) \(g V_i\)

- **Current-controlled voltage source**
  - \(-\) \(I_i\)
  - \(+\) \(r I_i\)

- **Current-controlled current source**
  - \(-\) \(I_i\)
  - \(+\) \(\alpha I_i\)

Here, the diamond symbol represents the source, which produces voltage or current proportional to a different voltage or current elsewhere in the circuit. Let’s make this more concrete by looking at an example:
In this case, the current in the dependent source is equal a constant \((g_m)\) times the voltage \(v_c\), which is defined elsewhere in the circuit. Just like an independent current source, this dependent source will produce the same current, regardless of what is connected to it (unless \(v_c\) changes – then the current will change appropriately). As we’ll see in future notes, dependent sources are useful tools for modeling many advanced circuit elements.

### 15.3 Superposition

In this section, we are going to look at circuits with multiple voltage or current sources. In particular, we would like to introduce a very useful idea in working with circuits of this type – superposition.

Let’s think back to our seven-step circuit analysis procedure. To solve for the currents and node potentials in a circuit, we set up a matrix problem of the form \(A\vec{x} = \vec{b}\) where \(\vec{x}\) contained the unknown currents and node potentials, \(\vec{b}\) contained the independent current and voltage sources, and \(A\) described the relationship between them. Since this matrix equation describes a real system, we know that there is a unique solution. Therefore, \(A\) is invertible:

\[
\vec{x} = A^{-1}\vec{b}
\]

This means that we can describe any current or node potential (i.e. any element of \(\vec{x}\)) as a linear combination of the independent current and voltage sources (the elements of \(\vec{b}\)). For example, consider a circuit with \(n\) independent sources voltage sources \(V_{s1}...V_{sn}\), and \(m\) independent current sources \(I_{s1}...I_{sm}\). An arbitrary node potential \(u_i\) (or equivalently, an arbitrary current \(i_i\)) can be written as

\[
u_i = \alpha_1V_{s1} + ... + \alpha_nV_{sn} + \beta_1I_{s1} + ... + \beta_mI_{sm}
\]

where the \(\alpha\)’s and \(\beta\)’s are coefficients from inverting \(A\). Since this equation is linear, we can calculate each term of this equation separately and then add them together at the end. For example, if we want to calculate the first term, \(\alpha_1V_{s1}\) we can set all of the other voltage and current sources to zero, then solve for \(u_i\). Repeating this for every source then adding the results is equivalent to calculating \(u_i\) with all of the sources present. However, splitting up the calculations can help us see simplifications and patterns that might be less obvious with all of the sources present.

This procedure is known as **superposition** and can be summarized as follows:

**For each independent source** \(k\) (either voltage source or current source)
- **Set all other independent sources to 0**
  - Voltage source: replace with a wire
  - Current source: replace with an open circuit

**Compute the circuit voltages and currents due to this source** \(k\)

**Compute** \(V_{out}\) **by summing the** \(V_{out,k}\)’s **for all** \(k\).
Now we ask the question: why does it make sense to replace voltage sources with wires? If we look at the I-V plot of a voltage source $V_S$, where $I$ is the current going through the voltage source, then the plot would be a vertical line:

![I-V plot of a voltage source](image)

Now if we want to zero out this voltage source, we are setting $V_S = 0$. Then the I-V plot is exactly the $y$-axis.

![I-V plot with $V_S = 0$](image)

What does this mean? This means that it allows any current to go through, however the voltage drop always remains zero. This is exactly what a wire element (sometimes called a short circuit) does.

Now let’s look at why we replace current sources with open circuits. If we plot the I-V graph of a current source $I_S$, we get the following:

![I-V plot of a current source](image)
What if we turn off the current source? Then the I-V graph becomes the x-axis, i.e., the line $I = 0$.

What does this mean? This means no matter what voltage you apply, there will be no current. This is equivalent to an open circuit.

Now let’s illustrate this idea on the circuit below, where we would like to figure out $V_{out}$.

We first compute the output voltage due to $V_1$ and hence source $V_2$ will be replaced with a wire:
We can recognize this as a voltage divider circuit, and therefore we know that $V_{\text{out},1} = \frac{R_2}{R_1 + R_2} V_1$. Next we compute the output voltage due to $V_2$ and hence source $V_1$ will be replaced with a wire.

We again recognize that this is just a voltage divider circuit and therefore we can see that $V_{\text{out},2} = \frac{R_1}{R_1 + R_2} V_2$.

Finally, to get the output voltage $V_{\text{out}}$ of the original circuit, we add the contributions from each voltage source $V_{\text{out}} = V_{\text{out},1} + V_{\text{out},2} = \frac{R_2}{R_1 + R_2} V_1 + \frac{R_1}{R_1 + R_2} V_2$.

As a side note, we can apply the idea of replacing elements with equivalent elements (e.g. replacing a $V = 0$ voltage source with a wire) to resistors as well. When do resistors have an equivalent representation? We will try and demonstrate this graphically. Recall that by Ohm’s law, the I-V graph across a resistor looks like
We know that the slope of the line is equal to $\frac{1}{R}$. What happens in the limit where $R$ trends towards infinity? Then the line becomes the x-axis, which corresponds to an open circuit as we’ve seen earlier. Now what happens in the limit where $R$ trends towards zero? The line becomes the y-axis, which corresponds to a wire.

To summarize, **zero voltage source and zero resistance are equivalent to wires (i.e. short circuits); zero current source and infinite resistance are equivalent to open circuits.**