Announcements

- Quest is next Monday! Logistics post will be up on Ed.
  
  → 2:10 → 3:10pm in Li Ka Shing 245
Agenda

- State Transition Systems
- Matrix Inverses
- Vector spaces
Setup: Review

\[
\hat{x}[t] = \begin{bmatrix} q_{a}[t] \\ q_{b}[t] \\ q_{c}[t] \end{bmatrix}
\]

\[
\hat{X}[t+1] = M \hat{X}[t]
\]
Setup

\[ M = \begin{bmatrix}
    P_{AA} & P_{BA} & P_{CA} \\
    P_{AB} & P_{BB} & P_{CB} \\
    P_{AC} & P_{BC} & P_{CC}
\end{bmatrix} \]

\[ P_{AB} = \text{Proportion moving from A to B} \]

\[
\begin{bmatrix}
q_a(t+1) \\
q_b(t+1) \\
q_c(t+1)
\end{bmatrix} = 
\begin{bmatrix}
P_{AA}q_a(t) + P_{BA}q_b(t) + P_{CA}q_c(t) \\
P_{AB}q_a(t) + P_{BB}q_b(t) + P_{CB}q_c(t) \\
P_{AC}q_a(t) + P_{BC}q_b(t) + P_{CC}q_c(t)
\end{bmatrix}
\]
Setup

\[ x[t+1] = M x[t] \]
\[ x[t+2] = M x[t+1] \]
\[ x[t+2] = M (M x[t]) \]
\[ x[t+2] = M^2 x[t] \]
\[ x[t] = M^t x[0] \]

Consider \( M = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \), what happens as \( t \to \infty \)?
Water Reservoirs: Example

\[
\begin{bmatrix}
0.5 & 0.5 & 0.5 \\
0.7 & 0.7 & 0.7 \\
0.2 & 0.2 & 0.2
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mu \quad \rho_a \quad \rho_c \\
\rho_a \quad \rho_a \quad \rho_a \\
\rho_c \quad \rho_c \quad \rho_c
\end{bmatrix}
\]

\[
M = [P_{aa} \quad P_{ab} \quad P_{ac}]
\]

\[
P_{ab} \quad \text{Proportion moving from A to B}
\]

If \( x[0] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \)

What is \( x[1] \)?
Conservation Property

A system is conservative iff the columns of the transition matrix sum to 1.

Are the following conservative?

\[
\begin{bmatrix}
0.2 & 0.3 & 1 \\
0.9 & 0.1 & 0 \\
0 & 0.6 & 0
\end{bmatrix}
\quad
\begin{bmatrix}
0.9 & 0.1 & 0 \\
0 & 0.2 & 0.8 \\
1 & 0 & 0
\end{bmatrix}
\]

Intuitive Meaning:
Water Reservoir: Example 2

Write out the transition matrix $M$. Is it conservative?
Moving Back in Time?

\[ x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} \]
Moving Back in Time?

Given $M$ and $\tilde{x}[t+1]$, can we find $\tilde{x}[t]$?

(i.e. can we find $B$ s.t. $B \times[t+1] = \times[t]$)

- Scalar Case:
  
  $\times[t+1] = m \times[t]$
Matrix Inversion

\[ M x[t] = x[t+1] \]

\[ \downarrow \]

\[ x[t] = M'^t x[t+1] \]
Inverse

Def. A square matrix $M$ is invertible if $\exists M^{-1}$ such that

$$M(M^{-1}) = (M^{-1})M = I$$
Uniqueness of Inverse

Thm. The inverse of a square matrix is unique.

Proof:
Finding the Inverse

$$MM^{-1} = I$$

known

known

$$\Leftarrow$$ How do we solve?
Example

\[
\begin{bmatrix}
1 & 1 & 1 & 0 \\
2 & 1 & 0 & 1
\end{bmatrix}
\]
Does an inverse always exist?

Think back to last lecture
Inverse: Equivalent Statements

Let $A$ be a square matrix, the following statements are equivalent:

1. $A$ is invertible
2. $A$ has linearly independent columns
3. $A$ has linearly independent rows
4. $A\overline{x} = \overline{b}$ has a unique solution for all $\overline{b}$
5. $A$ has a trivial null space
6. The determinant of $A \neq 0$

(next lectures)
Proof \[ \exists \mathbf{A}^{-1} \iff \text{lin. ind. columns} \]
Inverse of 2x2: Shortcut

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \]

\( ad-bc \) is the determinant of \( A \)
Vectorspaces

· **Def:** Set of vectors and scalars \((V, F)\) and two operators \(+, \cdot\) that satisfy the following:

1. \(\alpha \vec{x} \in V\)
2. \(\vec{x} + \vec{y} \in V\)
3. \(\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}\) (associativity)
4. \(\vec{x} + \vec{y} = \vec{y} + \vec{x}\) (commutativity)
5. \(\exists \vec{0} \text{ s.t. } \vec{x} + \vec{0} = \vec{x}\) (additive identity)
6. \(\exists (-\vec{x}) \text{ s.t. } \vec{x} + (-\vec{x}) = \vec{0}\) (additive identity)
7. \(\alpha(\vec{x} + \vec{y}) = \alpha\vec{x} + \alpha\vec{y}\) (distributivity)
8. \(\alpha(\beta \vec{x}) = (\alpha \beta)\vec{x}\) (multiplicative associativity)
9. \(\exists 1 \text{ s.t. } 1 \vec{x} = \vec{x}\)
10. \((\alpha + \beta)\vec{x} = \alpha\vec{x} + \beta\vec{x}\)
Vector Spaces: Examples