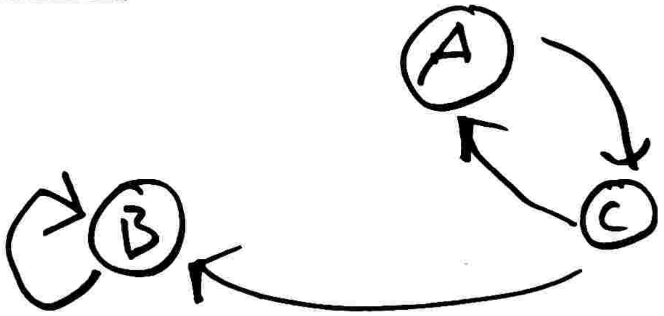
EECS 16A
2/6/20

• Matrix Inverse



$$\vec{x}(n+1) = A\vec{x}(n) \qquad \vec{x}(n) = \begin{bmatrix} x_A(n) \\ \vdots \\ x_C(n) \end{bmatrix}$$

By choosing $A \in \mathbb{R}^{n \times n}$, we introduce different ways of controlling/evolving the system over time.

E.g. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Consider 2 operations

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Longrightarrow \quad X_A(n+1) = X_A(n)$$

$$\vec{X}(n+1) = Q\vec{X}(n) = \begin{bmatrix} X_A(n+1) \\ X_B(n+1) \\ X_C(n+1) \end{bmatrix} = \begin{bmatrix} X_A(n) \\ X_C(n) \\ X_B(n) \end{bmatrix}$$

$$R = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} X_A(n+1) \\ X_B(n+1) \\ X_C(n+1) \end{bmatrix} = \begin{bmatrix} 1/2\, X_A(n) \\ 1/2\, X_B(n) \\ X_C(n) \end{bmatrix}$$

Suppose we start with $\vec{X}(1) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

run Q once, and then run R.

$$\vec{x}(3) = R\,\vec{x}(2) = R\left(Q\,\vec{x}(1)\right) = (RQ)\,\vec{x}(1)$$

$$\vec{x}(2) = Q\,\vec{x}(1)$$

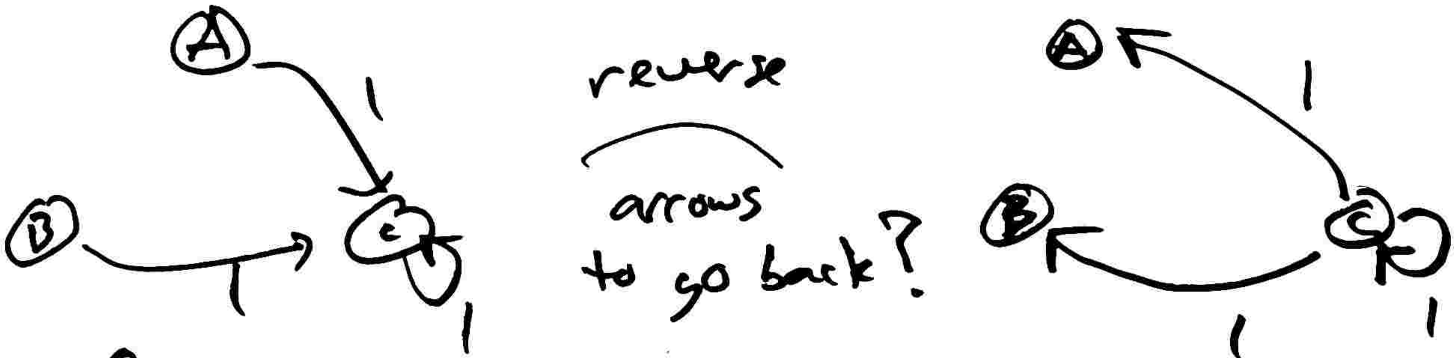
$$RQ = \cancel{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & & \\ & & \\ & & \end{bmatrix}}$$

$$\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\vec{x}(3) = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \\ 2 \end{bmatrix}$$

Now suppose $\vec{x}(n+1) = Q\,\vec{x}(n)$, $Q \in \mathbb{R}^{3\times 3}$

<u>Q</u>: How do we return to $\vec{x}(n)$ from $\vec{x}(n+1)$?



<u>Goal</u>: Find a matrix R such that:

$$\vec{x}(n) = R\,\vec{x}(n+1) = RQ\,\vec{x}(n)$$

$$\vec{x}(n+1) = Q\,\vec{x}(n) = QR\,\vec{x}(n+1).$$

$\Rightarrow$ R must satisfy $RQ = QR = I$.

<u>Def</u>: Let $P, Q \in \mathbb{R}^{n \times n}$. P is said to be the <u>inverse</u> of Q if $PQ = QP = I$.

<u>Note</u>: we denote inverse of Q by Q^{-1} (when it exists!)

<u>How to compute inverse of a matrix?</u>

<u>Goal</u>: want to solve $QP = \begin{bmatrix} | & | & & | \\ Q\vec{p}_1 & Q\vec{p}_2 & & Q\vec{p}_n \\ | & | & & | \end{bmatrix} = I$

$$\underbrace{Q\vec{p}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad Q\vec{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad Q\vec{p}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{n \text{ systems of LEQs in unknowns } \vec{p}_1 \cdots \vec{p}_n}$$

Do GE n times?

<u>observe</u>: GE of $A\vec{x} = \vec{b}$ does not depend on vector $\vec{b}$. So...

$$[\,Q \mid I\,] \xrightarrow{\text{reduce}} [\,\underbrace{Q_{rref}}_{I \text{ if inverse exists.}} \mid \overbrace{P}^{\text{inverse of } Q.}\,]$$

<u>Example</u>: $Q = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{swap}(R_1, R_3)} \left[\begin{array}{ccc|ccc} 1/2 & 0 & 0 & 0 & 0 & 1 \\ 1/2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

$R_1 \leftarrow 2R_1$

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix}$$

$$Q^{-1} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 2 \\ 1/2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

$R_2 \leftarrow R_2 - 1/2 R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

swap(R_2, R_3)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$\underbrace{}_{I} \qquad \underbrace{}_{Q^{-1}}$$

Question by Shreya: ~~If Q^{-1} is invertible~~,

does $(Q^{-1})^{-1} = Q$?

Def of inverse: P is inverse of Q^{-1}

if $PQ^{-1} = Q^{-1}P = I$.

Since Q^{-1} is inverse of Q, $QQ^{-1} = Q^{-1}Q = I$

$\Rightarrow$ Q is ~~an~~ the inverse of Q^{-1}

Question by Tom: If A is invertible,

then inverse of A is unique? A: Yes.

suppose there are two inverses B_1, B_2.

$$B_1 A = I$$

$$B_1 = B_1(\underbrace{A B_2}_{I}) = I B_2 = B_2$$

Example: $Q = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 6 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \leftarrow R_2 - 2R_1]{} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

$\Rightarrow$ solution does not exist

$\Rightarrow$ inverse does not exist.

The system of Eqns $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \vec{x} = \vec{0}$ has infinitely many solutions.

<u>Question by Shreyan #2</u>:

what happens if $A\vec{x} = \vec{b}$ has no solution and we try to invert?

<u>Thm</u>: If $\underset{\text{right-inverse}}{QP = I}$ and $\underset{\text{left-inverse}}{RQ = I}$

then $R = P$.

Stanford student solution:

$$\cancel{Q}P = R\cancel{Q} \Rightarrow P = R.$$

Berkeley Student solution:

$$P = \underbrace{RQ}_{I}P = RI = R.$$

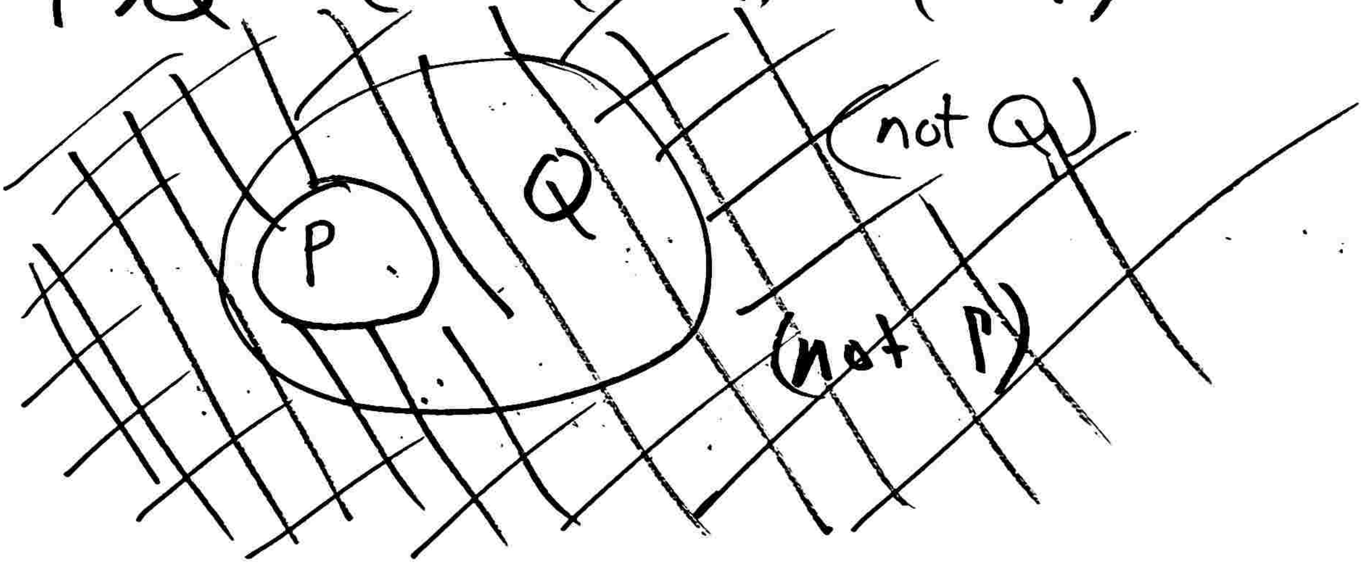
Generally speaking:

invertibility? $\longleftrightarrow$ unique solution in sys of LEQS? $\longleftrightarrow$ linear dependence of columns?

<u>Thm</u>: If the columns of A are linearly dependent, then A is not invertible.

[Equiv] <u>Thm</u>: If A is invertible, then cols. of A are linearly independent.

$P \Rightarrow Q \iff (\text{not } Q) \Rightarrow (\text{not } P)$



Pf: Suppose A^{-1} exists. Since cols of A linearly dependent $\underset{\text{def}}{\Longrightarrow}$ exists $\vec{x} \neq 0$ s.t. $A\vec{x} = \vec{0}$

$$\vec{x} = (A^{-1}A)\vec{x} = A^{-1}\vec{0} = \vec{0}$$