Last time: • Intro to Module 3 $M_{L,+}$
  • Inner Products

Today: More on Inner Products, with applications to Classification and Estimation. (TA: Amanda)

Recall: $\langle \cdot, \cdot \rangle$ is an inner product on a real vector space $V$ if:

1) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle \quad \forall \mathbf{u}, \mathbf{v} \in V$
2) $\langle \alpha \mathbf{u} + \mathbf{w}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle \quad \forall \alpha \in \mathbb{R}, \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$
3) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0 \quad \forall \mathbf{u} \in V$
   with equality only if $\mathbf{u} = \mathbf{0}$.

Induced Norm: $\| \mathbf{u} \| \overset{\text{def}}{=} \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$

defines a norm on $V$, i.e., $\| \cdot \|$ satisfies

1) $\| \alpha \mathbf{u} \| = |\alpha| \| \mathbf{u} \|$
2) $\| \mathbf{u} \| = 0 \iff \mathbf{u} = \mathbf{0}$
3) $\| \mathbf{u} + \mathbf{v} \| \leq \| \mathbf{u} \| + \| \mathbf{v} \|$.  

Ex: $\langle [3, 2] + [3], [y] \rangle = 3 \langle [2], [y] \rangle + \langle [3], [y] \rangle = 3 \langle [2], [y] \rangle + \langle [3], [y] \rangle$
Note: Even though $\| \cdot \|$ is not necessarily Euclidean norm, most of the intuition still applies.

E.g. Triangle Inequality: $\| \vec{u} + \vec{v} \| \leq \| \vec{u} \| + \| \vec{v} \|$. 

\[
\begin{pmatrix}
\text{Homework exercise using Cauchy-Schwarz Inequality:} \\
|\langle \vec{u}, \vec{v} \rangle| \leq \| \vec{u} \| \| \vec{v} \|
\end{pmatrix}
\]

E.g. Pythagorean Theorem

\[
\| \vec{u} + \vec{v} \|^2 = \| \vec{u} \|^2 + \| \vec{v} \|^2 \\
\]

= "perpendicular" in Euclidean geometry

= "orthogonal" for general inner products
**Def:** For inner product $\langle \cdot, \cdot \rangle$, two vectors $\vec{x}, \vec{y}$ are said to be orthogonal if $\langle \vec{x}, \vec{y} \rangle = 0$.

**Ex:** For Euclidean inner product

\[
\vec{x} \cdot \vec{y} = 0
\]

\[= ||\vec{x}|| \cdot ||\vec{y}|| \cdot \cos \Theta
\]

\[\implies \text{Euclidean norm, angle between } \vec{x}, \vec{y}
\]

\[\implies \vec{x}, \vec{y} \text{ are perpendicular (i.e. } \Theta = \frac{\pi}{2} \text{)}
\]

**Application to Classification**

**Problem:** Classify received signal to determine which satellite is transmitting.
\[ S_A = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \]

\[ S_B = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \]

Q: If I don't know what satellite is transmitting, how do I figure it out?

I observe \( \hat{r} = \hat{S}_i + \hat{\eta} \)

where \( i \in \{A, B\} \), and we want to determine which based on \( \hat{r} \).
Q: How to mathematically formulate this classification problem?

One method: minimize the "error" under $\|\cdot\|$.

Mathematically:

$$i^* = \arg \min_{i \in \{A, B\}} \| \hat{\mathbf{r}} - \hat{\mathbf{s}}_i \|^2$$

index I declare as which satellite was transmitting.

In words: choose satellite whose signature is closest to what I received.

$$\langle \hat{\mathbf{r}} - \hat{\mathbf{s}}_i, \hat{\mathbf{r}} - \hat{\mathbf{s}}_i \rangle = \langle \hat{\mathbf{r}}, \hat{\mathbf{r}} - \hat{\mathbf{s}}_i \rangle - \langle \hat{\mathbf{s}}_i, \hat{\mathbf{r}} - \hat{\mathbf{s}}_i \rangle$$

$$= \langle \hat{\mathbf{r}}, \hat{\mathbf{r}} \rangle - \langle \hat{\mathbf{r}}, \hat{\mathbf{s}}_i \rangle - \langle \hat{\mathbf{s}}_i, \hat{\mathbf{r}} \rangle + \langle \hat{\mathbf{s}}_i, \hat{\mathbf{s}}_i \rangle$$
\[ \text{fixed equal for each } i \text{ by design (under assumption of Euclidean norm).} \]

So, minimizing \( \| \hat{\mathbf{r}} - \hat{\mathbf{s}}_i \| \) over \( i \in \{ 1, 2 \} \) is equivalent to maximizing \( \langle \hat{\mathbf{r}}, \hat{\mathbf{s}}_i \rangle \) over \( i \in \{ 1, 2 \} \).

**Classification Procedure**

For \( i \in \{ 1, 2 \} \),

- compute \( \langle \hat{\mathbf{r}}, \hat{\mathbf{s}}_i \rangle \)

Return index \( i \) that maximizes this.

*Summary*: Inner products quantitatively capture notion of vector "similarity", which can be exploited in applications.
halfspace classifies as $B$ (translating)
if $z_b$ lies in this

halfspace classifies as $A$ (translating)
if $z_b$ lies in this
For $i = 1, \ldots, 24$
compute $<\hat{\mathbf{r}}, \hat{\mathbf{s}}_i>$.
Return maximum index $i$.

Two questions: 1) Interference.
   2) Timing.

Possibility 1: Both satellites transmitting.
Possibility 2: Only A transmitting.
Possibility 3: Only B transmitting.
Basic Procedure: Compute \( \langle \hat{r}, \hat{S}_i \rangle \)
to determine whether \( \hat{S}_i \) is present.
(i.e. satellite \( i \) is transmitting)

Assume henceforth \( \langle \cdot, \cdot \rangle = \text{Euclidean inner product} \)

Situation 1: \( \hat{r} = \hat{S}_A + \hat{S}_B + \hat{n} \).

\[
\langle \hat{r}, \hat{S}_A \rangle = \langle \hat{S}_A + \hat{S}_B + \hat{n}, \hat{S}_A \rangle \\
= \langle \hat{S}_A, \hat{S}_A \rangle + \langle \hat{S}_B, \hat{S}_A \rangle + \langle \hat{n}, \hat{S}_A \rangle \\
= 4 + [1 -1 1 -1] \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} + \text{"small"} \\
\approx 4
\]

\[
\langle \hat{r}, \hat{S}_B \rangle = \langle \hat{S}_A + \hat{S}_B + \hat{n}, \hat{S}_B \rangle \\
\approx 4
\]
Situation 2: \( \vec{r} = \vec{s}_A + \vec{n} \).

\[ \langle \vec{r}, \vec{s}_A \rangle \approx 0 \]

\[ \langle \vec{r}, \vec{s}_B \rangle = \langle \vec{s}_A + \vec{n}, \vec{s}_B \rangle \]

\[ = \langle \vec{s}_A, \vec{s}_B \rangle + \langle \vec{n}, \vec{s}_B \rangle \]

\[ \approx \text{“small”} \]

\[ \approx 0 \]

Situation 4: \( \vec{r} = \vec{n} \) (no satellite)

\[ \langle \vec{r}, \vec{s}_A \rangle \approx 0 \]

\[ \langle \vec{r}, \vec{s}_B \rangle \approx 0 \]

Big Idea: Design signatures to be orthogonal. This allows us to “separate out” interfering signals.
In cell systems you've probably heard the term "OFDM", "CDMA"

- orthogonal frequency multiplexing
- division

Second Big Question:

What about timing?

(i.e., how do I know when transmission started?)
To determine delay systematically, compute
\[ \langle [r[i], \ldots, r[i+3]], \hat{s}_A \rangle \]
for each \( i = -\infty \) to \( \infty \).
Def: For sequences $x[n]$, $y[n]$, $n \in \text{Integers}$

Define cross-correlation as:

$$\text{Corr}_{x,y}[k] = \sum_{i=-\infty}^{\infty} x[i] y[i-k].$$

Looks scary, but idea is just $\text{Corr}_{x,y}[k]$ is inner product between $x$ and $y$ "shifted by $k$".

Examples to come next time!