Lecture OD (6/23/21)

Announcements

- HW0 due tomorrow midnight. HW Party tomorrow 2-4pm in the Woz (4th floor Soda)
- Discussion OD is today (next session 4-5pm in Wheeler 130)
- Lab+OH start next week!
Agenda

- Tomography (cont.)
- Gaussian Elimination
Tomography

- Find values inside solid by using measurements from outside

1. \( m_1 = X_1 + X_2 \)
2. \( m_2 = X_2 + X_4 \)
3. \( m_3 = X_3 + X_4 \)
4. \( m_4 = X_1 + X \)
5. \( m_5 = X_1 + X_4 \)
6. \( m_6 = X_3 + X_2 \)

- Note: we'll need to find out how to get good measurements
Tomography + Lab

- We can generalize this to finding desired values under constrained measurements.
Gaussian Elimination: Setup

\[ f_1(x_1, \ldots, x_n) = b_1 \]
\[ f_k(x_1, \ldots, x_n) = b_k \]

\[ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix} \]

\[ x_1 = k_1 \]
\[ x_n = k_n \]
Operations

- Legal operations:
  1. Swap Equations
     \[ f_1(x_1, \ldots, x_n) = b_1 \Rightarrow f_2(x_1, \ldots, x_n) = b_2 \]
     \[ f_2(x_1, \ldots, x_n) = b_2 \Rightarrow f_1(x_1, \ldots, x_n) = b_1 \]
  2. Scale by non-zero scalar
     \[ f_1(x_1, \ldots, x_n) = b_1 \Rightarrow \alpha f_1(x_1, \ldots, x_n) = \alpha b_1 \]
  3. Add two equations
     \[ f_1(x_1, \ldots, x_n) = b_1 \Rightarrow f_1(x_1, \ldots, x_n) + f_2(x_1, x_n) = b_1 + b_2 \]
     \[ f_2(x_1, \ldots, x_n) = b_2 \]
Augmented Form

\[
\begin{bmatrix}
a_{11} & \cdots & a_{1m} \\
\vdots & \ddots & \vdots \\
a_{k1} & \cdots & a_{kn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix}
= 
\begin{bmatrix}
b_1 \\
\vdots \\
b_k
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
a_{11} & \cdots & a_{1m} & b_1 \\
\vdots & \ddots & \vdots & \vdots \\
a_{k1} & \cdots & a_{kn} & b_k
\end{bmatrix}
\]

1. We can convert to "augmented form" b/c the unknown vector does not change (remember its order though).

2. Rows represent equations so operations must be done to both sides of the bar.
Augmented Conversion

\[ \begin{align*}
0 \{ & \quad x + 2y + 3z = 2 \\
& \quad 2x - 3y = 1 \\
& \quad 2z + 3y + x = 2 - z \}
\end{align*} \]

\[ \Rightarrow \begin{bmatrix} 1 & 2 & 3 & | & 2 \\
0 & -1 & 3 & | & 1 \\
1 & 3 & 3 & | & 2 \end{bmatrix} \]

\[ \begin{align*}
0 & \Rightarrow x = 2 \\
y + z = 3
\end{align*} \]
Example

\[3x + 6y = 40 - 2x\]
\[61 = 9y + 8x\]

\[
\begin{bmatrix}
\frac{5}{6} & 6 & 40 \\
8 & 9 & 61
\end{bmatrix}
\Rightarrow
\frac{1}{5}
\begin{bmatrix}
1 & \frac{5}{6} & 8 \\
8 & 9 & 61
\end{bmatrix}
\]

Could have solved here:
\[x + \frac{5}{6}y = 8\]
\[-\frac{2}{3}y = -3\]
\[y = 5\]

\[61 - 64\]
Gaussian Elimination: Steps

1. Starting with the leftmost column with at least one non-zero value, pick an equation and scale it so that its non-zero value becomes a 1.

2. Move that equation as high as you can so the only rows above it have non-zero terms to the left of the column we're focusing on.

3. Use that equation to eliminate the rest of the values in that column below it.
4) Repeat steps 1-3 for the rest of the columns.

5) If there is a unique solution, the last row will now be an equation with only one unknown. Substitute that value into the equation of the previous row to get the value of the next unknown and repeat until solution is found.
Gaussian Elimination: Steps

\[ \begin{bmatrix} * & * & * & | & A_1 \\ * & * & * & | & A_2 \\ * & * & * & | & A_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & * & * & | & B_1 \\ 0 & * & * & | & B_2 \\ 0 & 0 & * & | & B_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & * & * & | & C_1 \\ 0 & 1 & * & | & C_2 \\ 0 & 0 & 1 & | & C_3 \end{bmatrix} \]

\[ x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \rightarrow x_1 = b_1 - a_{12}(b_2 - a_{23}b_3) - a_{13}b_3 \]
\[ x_2 + a_{23}x_3 = b_2 \rightarrow x_2 = b_2 - a_{23}b_3 \]
\[ x_3 = b_3 \]

This process is called "back substitution."

The rest of the steps are "row-reduction."
Example

\[
\begin{align*}
    x - y + 2z &= 1 \\
    2x + y + z &= 8 \\
    4x + 5y &= 7
\end{align*}
\]

\[
\begin{bmatrix}
    1&-1&2&|&1 \\
    2&1&1&|&8 \\
    -4&5&0&|&7
\end{bmatrix}
\]

\[
\begin{bmatrix}
    1&-1&2&|&1 \\
    0&3&-3&|&6 \\
    0&5&0&|&7
\end{bmatrix}
\]

\[
\begin{bmatrix}
    1&-1&2&|&1 \\
    0&0&1&|&2 \\
    0&0&0&|&9
\end{bmatrix}
\]

\[
\begin{bmatrix}
    1&0&-1&|&-2 \\
    0&1&1&|&3 \\
    0&0&1&|&2
\end{bmatrix}
\]

\[
\begin{bmatrix}
    1&0&-1&|&-2 \\
    0&1&1&|&3 \\
    0&0&1&|&2 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
    x = 2 \\
y = 3 \\
z = 1
\end{bmatrix}
\]
Jargon

row reduction

- Part of Gaussian elimination process where you're eliminating unknowns from a subset of the equations

back substitution

- Part of Gaussian elimination process where you've converted back to equation form and now are replacing unknowns with their known values

leading entry

- First non-zero entry in row
Jargon

Row Echelon Form

The matrix is in row echelon form iff:

1. All zero rows are below non-zero rows
2. The leading entry of a non-zero row is to the right of rows above
3. All leading entries are 1

\[ \begin{bmatrix} 1 & * & * & * & | & * \\
0 & 1 & * & * & | & * \\
0 & 0 & 0 & 0 & | & 0 \\
0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \]
Jargon

Reduced Row Echelon Form

- The matrix is in row reduced echelon form iff:
  1. It's in row echelon form
  2. Each leading entry is the only non-zero entry in its column

\[
\begin{bmatrix}
1 & 0 & * & 0 & \cdots & \ast \\
0 & 1 & * & 0 & \cdots & \ast \\
0 & 0 & 0 & 1 & \cdots & \ast \\
0 & 0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\]
Jargon

Pivots

- Leading entries in row echelon form

Basic Variables

- Unknowns corresponding to columns with pivots

Free variables

- Unknowns whose corresponding columns have no pivots
Jargon

Basic variables

Free variable

Pivots
More Examples

\[
\begin{align*}
2x + 4y + 2z &= 8 \\
x + y + z &= 6 \\
x - y - z &= 4
\end{align*}
\]

\[
\begin{bmatrix}
2 & 4 & 2 \\
1 & 1 & 1 \\
1 & -1 & -1
\end{bmatrix}
\]

\[
\begin{bmatrix}
2 \\
1 \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 1 \\
1 & -1 & -1 \\
0 & 1 & 2
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 1 & 4 \\
0 & 1 & 2 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 4 \\
0 & 1 & 0 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 1 & 4 \\
0 & 1 & 2 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 1 & 4 \\
0 & 1 & 2 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
More Examples

\[
\begin{align*}
\pi + 2y + z + w &= 2 \\
y + z &= 0 \\
2x + 3y + z &= 4
\end{align*}
\]
More examples

\[
\begin{align*}
\alpha + 4\beta + 2\gamma &= 2 \\
\alpha + 2\beta + 8\gamma &= 0 \\
\alpha + 3\beta + 5\gamma &= 3
\end{align*}
\]

\[
\begin{bmatrix}
1 & 4 & 2 \\
0 & -2 & 6 \\
0 & -1 & 3
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 4 & 2 & | & 2 \\
1 & 2 & 8 & | & 0 \\
1 & 3 & 5 & | & 3
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 4 & 2 \\
0 & -2 & 6 \\
0 & -1 & 3
\end{bmatrix}
\begin{bmatrix}
1 & 4 & 2 \\
0 & -1 & 3 \\
0 & -3 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & | & 2
\end{bmatrix}
\]

\( \Delta = 2 \)
How many solutions?

1. If a system has a row of all zeros, how many solutions does it have?
   - Infinite
   - No Solution
   - Unique

2. If a system has a pivot in every row, how many solutions does it have?
   - Infinite
   - Unique
How many Solutions?

3. If a system has a row where the leading entry is in the output vector, how many solutions does it have?

\[
\begin{bmatrix}
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

No Solution

4. What do rows of zeros tell us about our initial set of equations?

Redundancy in equation
How many solutions?

8. What do we need to be true for there to be a unique solution?
   (2 things)
   - Pivot in every column
   - No inconsistent rows