EECS 16A
Lec 0B, 1/23/20.

Tomography Example.

\begin{align*}
x_1 + x_2 &= b_1 \\
x_3 + x_4 &= b_2 \\
x_1 + x_3 &= b_3 \\
x_2 + x_4 &= b_4 \\
x_1 + x_4 &= b_5
\end{align*}

Assume \( b_1 = b_2 = b_3 = b_4 = 1 \)

System of Linear Equations

\begin{align*}
e^{x_1} + \log x_2 &= 5 \\
\sin(x_1) + 10 \cdot (x_2) &= 72
\end{align*}

Linear Algebra = tools for working with linear equations.

\( x_1 = x_2 = x_3 = x_4 = \frac{1}{2} \).

\( x_2 = x_3 = 1 \quad x_1 = x_4 = 0 \).
Linear equation: \[ f(x_1, \ldots, x_n) = b \quad b \in \mathbb{R} \]

\[ f: \mathbb{R}^n \rightarrow \mathbb{R} \]

\[ f = \text{linear function}. \]

**Def:** A function \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) is linear if:

\[
\begin{align*}
  f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \ldots, \alpha x_n + \beta y_n) &= \alpha f(x_1, \ldots, x_n) \\
  &\quad + \beta f(y_1, \ldots, y_n)
\end{align*}
\]

\[ \forall \alpha, \beta, x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}. \]

**Fact:** Every linear function \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) has the form

\[ f(x_1, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \]

where \( a_i \)'s = \text{coefficients} (\text{just numbers not depend on } x_i \text{'s}), \ x_i \text{'s = variables}. \]
Why?

\[ f(x_1, \ldots, x_n) = f(x_1, 1 + \frac{1}{x_2}, x_0 + \frac{1}{x_3}, \ldots, x_0 + \frac{1}{x_n}) \]
\[ = x_1 f(1, 0, \ldots, 0) + f(0, x_2, x_3, \ldots, x_n) \]
\[ = x_1 f(1, 0, \ldots, 0) + x_2 f(0, 1, 0, \ldots, 0) + \ldots + x_n f(0, 0, \ldots, 1) \]
\[ = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n. \]

So, a system of linear EQns. has the form:
\[ a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1 \quad \text{\textit{a}}_{ij}'s = \textit{coefficients (design)} \]
\[ a_{21} x_1 + a_{22} x_2 + \ldots = b_2 \quad \textit{x}_i's = \textit{variables (unknowns)} \]
\[ \vdots \quad \vdots \]
\[ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = b_m \quad \textit{b}_i's = \textit{constants (measurements)} \]
Shorthand notation for expressing the system of equations:

\[
\begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
  a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{bmatrix}
\]

augmented matrix

Ex: Tomography

\[
\begin{bmatrix}
  1 & 1 & 0 & 0 & b_1 \\
  0 & 0 & 1 & 1 & b_2 \\
  1 & 0 & 1 & 0 & b_3 \\
  0 & 1 & 0 & 1 & b_4 \\
  0 & 0 & 0 & 1 & b_5
\end{bmatrix}
\]

Question: How to systematically solve systems of LEQs?
Ex:
\[
\begin{align*}
2x + 3y &= 8 \\
3x - y &= 1
\end{align*}
\]
\[
\begin{align*}
\downarrow x + \frac{3}{2}y &= 4 \\
3x - y &= 1
\end{align*}
\]
\[
\begin{align*}
\downarrow x + \frac{3}{2}y &= 4 \\
\quad - \frac{1}{2}y &= -11
\end{align*}
\]
\[
\begin{align*}
\downarrow x + \frac{3}{2}y &= 4 \\
\quad y &= 2
\end{align*}
\]
\[
\begin{align*}
\downarrow x &= 1 \\
\quad y &= 2
\end{align*}
\]
Operations at each step:
- Rescale a row by multiplying by non-zero scalar
- Added to any row, a scalar multiple of any other row
- Swap any two rows.

These are called elementary row operations. When Gaussian Elimination: algorithm for solving a system of equations by performing elementary row operations on the augmented matrix. How does it go? You already did it!

Def: The leading entry of a row is the first non-zero entry in that row.

\[
\begin{bmatrix}
1 & 3/2 & 4 \\
0 & -11/2 & -11
\end{bmatrix}
\]
Gaussian Elimination:

Step 1: For $i = 1 \cdots m$

1. If necessary, swap row $i$ with a row below it so that the leading entry in row $i$ is as far left as possible.

2. Scale row $i$ so that leading entry is 1.

3. For $j = i+1 \cdots m$, add to row $j$ a scalar multiple of row $i$ so that the leading entry in row $i$ has all zeros below it.

Observation: Step 1 will result in an augmented matrix of the form

\[
\begin{bmatrix}
1 & * & * & * \\
0 & 1 & * & * \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \\
\]

3 Properties:
1) All-zero rows are at bottom.
2) For each leading entry (pivot), all entries to left and/or below = 0
3) All leading entries = 1.
Step 2: Back-substitution

For $i = m \ldots 1$, Add to each row $j = 1 \ldots i-1$ a multiple of row $i$, so that leading entry of row $i$ has all zeros above it.

Observe: Leaves us with a matrix of form:

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 & \ast \\
0 & 1 & \cdots & 0 & \ast \\
0 & 0 & \cdots & 1 & \ast \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}
\]

Such a matrix satisfies 2 properties:

1) It is in row-echelon form
2) Each leading entry is the only non-zero entry in its column

This is called reduced row-echelon form (rref)

Fact: Each augmented matrix has a unique corresponding matrix in rref.
Once augmented matrix is reduced to rref, variables corresponding to columns containing pivots (leading entries) are called **basic variables**. Other variables are called **free variables**.

**Ex:**

\[
\begin{align*}
2x + 3y &= 8 \\
2x + 3y &= 6
\end{align*}
\]

\[
\begin{bmatrix}
2 & 3 & | & 8 \\
2 & 3 & | & 6 \\
\end{bmatrix}
\]

\[
R_2 \leftarrow R_2 - R_1
\]

\[
\begin{bmatrix}
2 & 3 & | & 8 \\
0 & 0 & | & -2 \\
\end{bmatrix}
\]

\[
R_1 \leftarrow R_1 / 2
\]

\[
\begin{bmatrix}
1 & 3/2 & | & 4 \\
0 & 0 & | & -2 \\
\end{bmatrix}
\]

impossible \(\rightarrow\) no solution
Ex: \[ x + 4y = 6 \]
\[ 2x + 8y = 12 \]

basic variable   free variable
\[ y \]   \[ x \]

infinitely many solutions. To obtain a solution, select free variable freely

\[ y = 0 \quad x = 6 \]

If a system of LEQs has a solution; consistent

\[ \begin{bmatrix} 1 & 4 & 6 \\ 2 & 8 & 12 \end{bmatrix} \]

\[ R_2 \leftarrow R_2 - 2R_1 \]

\[ \begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 0 \end{bmatrix} \]