This homework is due April 10, 2020, at 23:59.
Self-grades are due April 13, 2020, at 23:59.

Submission Format
Your homework submission should consist of one file.

- hw10.pdf: A single PDF file that contains all of your answers (any handwritten answers should be scanned)

Submit each file to its respective assignment on Gradescope.

1. Op-Amp in Negative Feedback
In this question, we analyze op amp circuits that have finite gain. We replace the op amp with its circuit model with parameterized gain and observe the gain’s effect on terminal and output voltages as the gain approaches infinity. Figure 1 shows the equivalent model of the op-amp.

![Figure 1: Op-amp model](image)

(a) Consider the circuit shown in Figure 1. Assume that the op amp is ideal ($A \to \infty$) for parts (a) through (e). What is $u_+ - u_-$?

**Solution:** For ideal op amp circuits in negative feedback, the voltage at the two terminals must be equal, so $u_+ - u_- = 0$.

(b) Find $v_x$ as a function of $v_{out}$.

**Solution:** We see that $v_x$ is the middle node of a voltage divider, so $v_x = v_{out} \frac{R_1}{R_1 + R_2}$. 
(c) What is the current flowing through $R_2$ as a function of $v_s$?

**Solution:** We know from part (a) that $v_x = v_s$. The current flowing through $R_1$ is $I_{R_1} = \frac{v_s}{R_1}$. This current also flows through $R_2$.

(d) Find $v_{out}$ as a function of $v_s$.

**Solution:** Using the answer from the previous part, $v_{out} = v_s + R_2 I_{R_1} = v_s + R_2 \frac{v_s}{R_1} = v_s \left( 1 + \frac{R_2}{R_1} \right)$.

(e) What is the current $i_L$ through the load resistor $R$? Give your answer in terms of $v_{out}$.

**Solution:** The current $i_L$ through the load is $\frac{v_{out}}{R}$.

(f) Draw an equivalent circuit by replacing the op-amp with the op-amp model shown in Figure 1 and calculate $v_{out}$ and $v_x$ in terms of $A$, $v_s$, $R_1$, $R_2$ and $R$. Is the magnitude of $v_x$ larger or smaller than the magnitude of $v_s$? Do these values depend on $R$?

**Solution:**
This is the equivalent circuit of the op-amp:
Since \( v_{\text{out}} \) is connected to the output of the op-amp, which is a voltage source, we can determine \( v_{\text{out}} \):

\[
v_{\text{out}} = A(u_+ - u_-)
\]

\[
= A(v_s - v_x)
\]

Since there is no current flowing into the op amp input terminals from nodes \( u_+ \) and \( u_- \), \( R_1 \) and \( R_2 \) form a voltage divider and \( v_x = v_{\text{out}} \left( \frac{R_1}{R_1 + R_2} \right) \). Thus, substituting and solving for \( v_{\text{out}} \):

\[
v_{\text{out}} = A \left( v_x - v_{\text{out}} \frac{R_1}{R_1 + R_2} \right)
\]

\[
v_{\text{out}} = v_s \left( \frac{1}{\frac{R_1}{R_1 + R_2} + \frac{1}{A}} \right)
\]

Knowing \( v_{\text{out}} \), we can find \( v_x \):

\[
v_x = \frac{v_s}{1 + \frac{R_1 + R_2}{A R_1}}
\]

Notice that \( v_x \) is slightly smaller than \( v_s \), meaning that in equilibrium in the non-ideal case, \( u_+ \) and \( u_- \) are not equal. \( v_{\text{out}} \) and \( v_x \) do not depend on \( R \), which means that we can treat \( v_{\text{out}} \) as a voltage source that supplies a constant voltage independent of the load \( R \).

(g) Using your solution to the previous part, calculate the limits of \( v_{\text{out}} \) and \( v_x \) as \( A \to \infty \). Do you get the same answer as in part (d)?

**Solution:**

As \( A \to \infty \), the fraction \( \frac{1}{A} \to 0 \), so

\[
v_{\text{out}} = v_s \left( \frac{1}{\frac{R_1}{R_1 + R_2} + \frac{1}{A}} \right)
\]

converges to

\[
v_s \left( \frac{1}{\frac{R_1}{R_1 + R_2} + 0} \right) = v_s \left( \frac{R_1 + R_2}{R_1} \right)
\]

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Therefore, the limits as \( A \to \infty \) are:

\[
v_{\text{out}} \to v_s \left( \frac{R_1 + R_2}{R_1} \right) \\
v_x \to v_s
\]

If we observe the op amp is in negative feedback, we can apply the fact that \( u_+ = u_- \). We get \( v_x = v_s \). Then the current \( i \) flowing through \( R_1 \) to ground is \( \frac{v_{\text{out}} - v_x}{R_1} \). By KCL, this same current flows through \( R_2 \) since no current flows into the negative input terminal of the op amp (\( u_- \)). Thus, the voltage drop across \( R_2 \) is \( v_{\text{out}} - v_x = i \cdot R_2 = v_s \left( \frac{R_2}{R_1} \right) \). Therefore, \( v_{\text{out}} = v_s + v_s \left( \frac{R_2}{R_1} \right) = v_s \left( \frac{R_1 + R_2}{R_1} \right) \). The answers are the same if you take the limit as \( A \to \infty \).

(h) Now you want to make a circuit whose gain is nominally \( G_{\text{nom}} = \frac{v_{\text{out}}}{v_s} = 2 \) with a minimum error of 1\% (a minimum gain of \( G_{\text{min}} = 1.98 \)). What is the minimum required gain of the amplifier \( A_{\text{min}} \) to achieve that specification?

**Solution:** From the previous part, \( v_{\text{out}} = v_s \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{A} \right) \). After algebraic manipulations, we get

\[
v_{\text{out}} = v_s \left( \frac{A(R_1 + R_2)}{R_1 + R_2 + AR_1} \right)
\]

We are interested in the op amp’s minimum gain \( A_{\text{min}} \), which gives us the circuit’s corresponding minimum gain \( G_{\text{min}} \). Solving for \( A \) and substituting in the minimum values \( A_{\text{min}} \) and \( G_{\text{min}} \) gives:

\[
A_{\text{min}} = \frac{G_{\text{min}}(R_1 + R_2)}{R_1 + R_2 - G_{\text{min}}R_1}
\]

Rewriting \( A_{\text{min}} \) in terms of \( G_{\text{nom}} = 1 + \frac{R_2}{R_1} \) gives:

\[
A_{\text{min}} = \frac{G_{\text{min}}G_{\text{nom}}}{G_{\text{nom}} - G_{\text{min}}} = 198
\]

Notice that the op amp’s minimum gain is independent of the resistor values. In general, if we wanted an error of less than \( \varepsilon \), then the following will approximately hold: \( \frac{A_{\text{min}}}{G_{\text{nom}}} > \frac{1}{\varepsilon} \).

2. **PetBot Design**

In this problem, you will design circuits to control PetBot, a simple robot designed to follow light. PetBot measures light using photoresistors. A photoresistor is a light-sensitive resistor. As it is exposed to more light, its resistance decreases. Given below is the circuit symbol for a photoresistor.

![Photoresistor Symbol]

Below is the basic layout of the PetBot. It has one motor on each wheel. We will model each motor as a 1\( \Omega \) resistor. When motors have positive voltage across them, they drive forward; when they have negative voltage across them, they drive backward. At zero voltage across the motors, the PetBot stops. The speed of the motor is directly proportional to the magnitude of the motor voltage. The light sensor is mounted to the front of the robot.
(a) **Speed control** – Let us begin by first having PetBot decrease its speed as it drives toward the flashlight. Design a motor driver circuit that outputs a decreasing positive motor voltage as the PetBot drives toward the flashlight. The motor voltage should be at least 5 V far away from the flashlight. When far away from the flashlight, the photoresistor value will be 10 kΩ and dropping toward 100 Ω as it gets closer to the flashlight.

In your design, you may use any number of resistors with any value and just 1 op-amp. You also have access to voltage sources of 10 V and −10 V. Based on your circuit, derive an expression for the motor voltage as a function of the circuit components that you used.

**Hint 1:** You should consider the loading effect of connecting this circuit to your motor, which has resistance. A buffer may help solve this problem.

**Hint 2:** If you’re not sure where to start, try playing around with connecting the circuit elements in different ways, and think about circuits you have seen before.

**Solution:**

We can use a voltage divider circuit to adjust the output motor voltage as the PetBot drives towards the flashlight (and the photoresistor’s resistance decreases).
The output of the above circuit is:

\[ v_{out} = \frac{R_p}{R_p + R} \cdot 10V \]

where \( R_p \) represents the photoresistor. Note that we use a voltage buffer to prevent loading effects when connecting the motor.

We set \( R \leq 10k\Omega \) to achieve \( v_{out} \geq 5V \) when the PetBot is far away from the flashlight (i.e. \( R_p = 10k\Omega \)). As the PetBot drives towards the flashlight, the resistance \( R_p \) drops, so that \( v_{out} \), the motor voltage, decreases.

(b) **Distance control** – Let us now have PetBot drive up to a flashlight (or away from the flashlight) and stop at distance of 1 m away from the light. At the distance of 1 m from the flashlight, the photoresistor has a value 1 kΩ.

Design a circuit to output a motor voltage that is positive when the PetBot is at a distance greater than 1 m from the flashlight (making the PetBot move towards it), zero at 1 m from the flashlight (making the PetBot stop), and negative at a distance of less than 1 m from the flashlight (making the PetBot back away from the flashlight.)

In your design, you may use any number of resistors of any value and just 1 op-amp. You also have access to voltage sources of 10V and \(-10V\). Based on your circuit, derive an expression for the motor voltage as a function of the values of circuit components that you used.

![Circuit Diagram]

**Solution:**

We outline two possible solutions here:

**Method 1:**

Here observe that \( v_{out} = u_+ \). Using superposition we find that

\[ v_{out} = u_+ = 10V \frac{R_p}{R + R_p} - 10V \frac{R}{R + R_p} = 10V \frac{R_p - R}{R + R_p} \]

To satisfy the condition that \( v_{out} = 0V \) when PetBot is 1 m away, we have that \( R = 1 k\Omega \). Similar to the previous design, we can do the analysis for when the Petbot is far away and close by. We will show how to do it when the Petbot is close by here.

\[ R_p < R = 1 k\Omega \]
Method 2:

Choosing $R_1 = R = 1 \text{k} \Omega$ we observe that the voltage $v_{out} = 0V$ when Petbot is 1 m away as required.

To find $v_{out}$ more generally, observe that $u_- = u_+ = 0$, so we need to find the current $i$ going through $R_2$ from node $u_-$ to node $u_{out}$ to get $v_{out}$.

$$v_{out} = -iR_2 = -\left(\frac{10V}{R_p} + \frac{-10V}{R_1}\right)R_2$$

From this we see that when PetBot is greater than 1 m away we have

$$R_p > R_1 = 1 \text{k} \Omega$$

$$\frac{1}{R_p} - \frac{1}{R_1} < 0$$

$$\Rightarrow -\left(\frac{1}{R_p} - \frac{1}{R_1}\right) \cdot R_2 10V > 0$$

Similarly when the Petbot is less than 1 m away, we have that the motor voltage will be negative.

3. **Rain Sensor v2.0**

In a previous homework, we analyzed a rain sensor built by a lettuce farmer in Salinas Valley. They used a rectangular tank outside and attached two metal plates to two opposite sides in an effort to make a capacitor whose capacitance varies with the amount of water inside. The width and length of the tank are both $w$ (i.e. the base is square), and the height of the tank is $h_{tot}$. 
As your EE16A circuits toolkit is now complete with capacitors, op-amps, and switches, we will revisit this problem to improve the readout electronics. The goal is to create a circuit block that will output voltage as a linear function of the water height, $h_{H_2O}$.

(a) What is the capacitance between terminals $a$ and $b$ when the tank is empty, $C_{empty}$? Again, the height of the water in the tank is $h_{H_2O}$. Modeling the tank as a pair of capacitors in parallel, find the total capacitance $C_{tank}$ between the two plates. Can you write $C_{tank}$ as a function of $C_{empty}$?

*Note:* The permittivity of air is $\varepsilon$, and the permittivity of rainwater is $81\varepsilon$.

**Solution:**

$$C_{empty} = \frac{\varepsilon_{air} h_{tot} w}{w} = \varepsilon h_{tot}$$

For $C_{tank}$, we can break the total capacitance into two parts. First, let’s calculate the capacitance of the two plates separated by water:

$$C_{water} = \frac{\varepsilon_{H_2O} h_{H_2O} w}{w} = 81\varepsilon h_{H_2O}$$

And now, we can calculate the capacitance of the two plates separated by air:

$$C_{air} = \frac{\varepsilon_{air} (h_{tot} - h_{H_2O}) w}{w} = \varepsilon (h_{tot} - h_{H_2O})$$

Because these two capacitors appear in parallel, we can simply add our two previous results to find the total equivalent capacitance:

$$C_{tank} = C_{water} + C_{air} = \varepsilon (h_{tot} + 80h_{H_2O})$$

Now, we can rewrite the above equation to show the dependence on $C_{empty}$:

$$C_{tank} = C_{empty} + 80\varepsilon h_{H_2O}$$

(b) Here, we will analyze a circuit that transfers all charges for efficient readout. For the circuit below, draw the output waveform of $v_{out}$ as a function of $v_{in}$, $C_f$, and $C_{in}$. 

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Solution:
In $\phi_1$, $C_{\text{in}}$ gets charged up to 

$$Q_{C_{\text{in}},1} = C_{\text{in}}v_{\text{in}}.$$ 

On the other hand, since $v_{\text{out}}$ and $v_-$ are at zero potential, no charge is stored on $C_f$, so 

$$Q_{C_f,1} = 0.$$ 

In $\phi_2$, the positive plate of $C_{\text{in}}$ is shorted to ground, which forces all of the negative charges to move to the left plate of $C_f$. Now using charge conservation, 

$$Q_{C_{\text{in}},1} + Q_{C_f,1} = Q_{C_{\text{in}},2} + Q_{C_f,2}$$

$$C_{\text{in}}v_{\text{in}} + 0 = 0 + C_fv_{\text{out}}$$

$$v_{\text{out}} = \frac{C_{\text{in}}}{C_f}v_{\text{in}}$$
(c) We examined the non-inverting configuration in the previous part- now, we will look into the inverting configuration. For the circuit below, draw the output waveform of $v_{out}$ as a function of $v_{in}$, $C_f$, and $C_{in}$. 
Solution:
In $\phi_1$, the voltage across both the capacitors is zero, so

$$Q_{C_{in},1} = Q_{C_f,1} = 0.$$ 

In $\phi_2$, $C_{in}$ gets charged up to $Q_{C_{in},2} = C_{in}v_{in}$. Since $C_f$ is in series, $C_f$ stores the same amount of charge but with opposite polarity, so $Q_{C_f,2} = -C_f v_{out}$. 

$$Q_{C_{in},2} = Q_{C_f,2}$$
$$C_{in}v_{in} = -C_f v_{out}$$
$$v_{out} = -\frac{C_{in}}{C_f}v_{in}$$

(d) With the help of the basic circuit blocks shown in parts (b) and (c), we will now implement a circuit that will output voltage as a linear function of the water height, $h_{H_2O}$. In addition to the rain-sensing capacitor, we will use two fixed value capacitors $C_{fixed}$ and $C_{empty}$. Use the values obtained in part (a) for $C_{tank}$ and $C_{empty}$. For the circuit below, draw the output waveform of $v_{out}$ as a function of $v_{in}$, $C_{fixed}$, $\varepsilon$, and $h_{H_2O}$.
**Solution:**

\[
\begin{align*}
V_{out} &= \left( \frac{C_{\text{tank}}}{C_{\text{fixed}}} - \frac{C_{\text{empty}}}{C_{\text{fixed}}} \right) V_{in} \\
&= \frac{C_{\text{empty}} + 80\varepsilon h_{H_2O} - C_{\text{empty}}}{C_{\text{fixed}}} V_{in} \\
&= \frac{80\varepsilon h_{H_2O}}{C_{\text{fixed}}} V_{in}
\end{align*}
\]
4. (PRACTICE) More Current Sources And Capacitors

For the circuit given below, give an expression for $v_{out}(t)$ in terms of $I_s$, $C_1$, $C_2$, $C_3$, and $t$. Assume that all capacitors are initially uncharged, i.e. the initial voltage across each capacitor is 0V.

\[
\text{Solution:}
\]

Instead of finding $v_{out}$ directly, let’s first find the voltage $v_{C_{eq}}$ across $C_2$ and $C_3$.

To do this, we replace $C_2$ and $C_3$ with their equivalent capacitance $C_{eq} = C_2 \parallel C_3 = \frac{C_2 C_3}{C_2 + C_3}$. 
We know that to solve for $v_{Ceq}$, we can find the equivalent capacitance of $C_1$ and $Ceq$ first, which is $C_1 + Ceq$. Since the capacitors are initially uncharged, $v_{Ceq}(0) = 0$.

$$v_{Ceq}(t) = \int \frac{I_s}{C_1 + Ceq} dt = \frac{I_t}{C_1 + Ceq} + v_{Ceq}(0) = \frac{I_t}{C_1 + Ceq}$$

Now that we know that voltage across the equivalent capacitor $Ceq$, we can find the current flowing through the equivalent capacitor $Ceq$.

$$i_{Ceq}(t) = Ceq \frac{dv_{Ceq}(t)}{dt} = \frac{CeqI_s}{C_1 + Ceq}$$

Note that the current $i_{Ceq}$ is equal to the current flowing through $C_3$ since $C_2$ and $C_3$ were originally connected in series.

$$i_C(t) = i_{Ceq}(t) = \frac{CeqI_s}{C_1 + Ceq}$$

Since $v_{out}$ is the voltage across the capacitor $C_3$, we integrate to find $v_{out}$. Again, since all capacitors are initially uncharged, $v_{out}(0) = 0$.

$$v_{out}(t) = \int \frac{CeqI_s}{C_3(C_1 + Ceq)} dt = \frac{CeqI_s}{C_3(C_1 + Ceq)} + v_{out}(0) = \frac{C_2C_3}{C_3(C_1 + Ceq)} \frac{I_t}{C_3} = \frac{C_2I_t}{C_1C_2 + C_1C_3 + C_2C_3}$$

5. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID’s. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.