EECS 16A Designing Information Devices and Systems I
Spring 2020 Homework 2

This homework is due Friday, February 7, 2020, at 23:59.
Self-grades are due Monday, February 10, 2020, at 23:59.

Submission Format

Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

Please attach a PDF of your Jupyter notebook for all the problems that involve coding. Make sure the results of your plots (if any) are visible. Please assign the PDF of the notebook to the correct problems on Gradescope — we will be unable to grade the problems without this assignment or submission.

1. Mechanical Gaussian Elimination and Linear Independence

Consider the following system of linear equations:

$$\begin{aligned} 4x+4y+4z+w+v &= 1 \\ x+y+2z+4w+v &= 2 \\ 5x+5y+5z+w+v &= 0 \end{aligned}$$

(a) Find the RREF of the augmented matrix of the system of linear equations.

(b) Which variables are the basic variables? Which variables are the free variables?

(c) Parameterize the solutions to the system of equations in terms of the free variables.

(d) State if the following sets of vectors are linearly independent or dependent. If the set is linearly dependent, provide a linear combination of the vectors that sum to the zero vector.

i $\left\{ \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$

ii $\left\{ \begin{bmatrix} -1 \\ 5 \\ 0 \\ -3 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -7 \\ 20 \\ 24 \\ -12 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ -4 \\ 0 \\ \frac{1}{3} \end{bmatrix} \right\}$

iii $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

iv $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

2. Finding Charges from Potential Measurements

We have three point charges Q_1, Q_2, and Q_3 whose positions are known, and we want to determine their charges. In order to do that, we take three electric potential measurements U_1, U_2 and U_3 at three different locations. You do not need to understand electric potential to do this problem. The locations of the charges and potentials are shown in Figure 1.

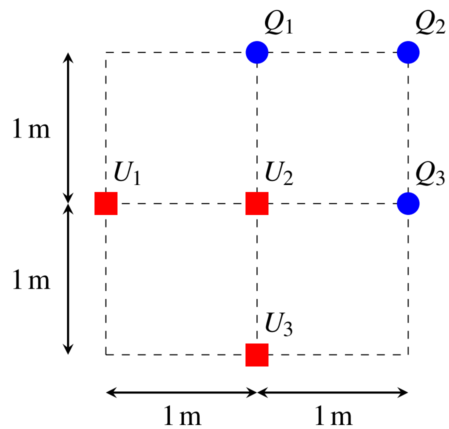


Figure 1: Locations of the charges and potentials.

For the purpose of this problem, the following equation is true:

$$U = k\frac{Q}{r}$$

at a point r meters away (for some fixed physical constant k; this problem does not require the numerical value of k).

Furthermore, the potential contributions from different point charges add up linearly. For example, in the setup of Figure 1, the potential measured at point U_2 is

$$U_2 = k\frac{Q_1}{1} + k\frac{Q_2}{\sqrt{2}} + k\frac{Q_3}{1}.$$

Given that the actual potential measurements in the setup of Figure 1 are

$$U_1 = k\frac{4+3\sqrt{5}+\sqrt{10}}{2\sqrt{5}},$$

$$U_2 = k\frac{2+4\sqrt{2}}{\sqrt{2}},$$

$$U_3 = k\frac{4+\sqrt{5}+3\sqrt{10}}{2\sqrt{5}},$$

write the system of linear equations relating the potentials to charges. Solve the system of linear equations to find the charges Q_1, Q_2, Q_3. You may use your IPython notebook to solve the system.

IPython hint: For constants a_i, b_i, c_i, y_i, you can solve the system of linear equations

$$a_1x_1 + a_2x_2 + a_3x_3 = y_1$$

$$b_1x_1 + b_2x_2 + b_3x_3 = y_2$$

$$c_1x_1 + c_2x_2 + c_3x_3 = y_3$$

in IPython with the following code:

```
import numpy as np
a = np.array([
    [a1, a2, a3],
    [b1, b2, b3],
    [c1, c2, c3]
])
b = np.array([y1, y2, y3])
x = np.linalg.solve(a, b)
print(x)
```

The square root of a number a can be written as `np.sqrt(a)` in IPython.

3. Kinematic Model for a Simple Car

Learning Goal: Many real world systems are not actually linear and have more complex behaviors. However, linear models can approximate non linear systems under certain conditions.

Building a self-driving car first requires understanding the basic motions of a car. In this problem, we will explore how to model the motion of a car.

There are several models that we can use to model the motion of a car. Assume we use the following kinematic model, given in the following four equations and Figure 2.

$$x[k+1] = x[k] + v[k]\cos(\theta[k])\Delta t \tag{1}$$
$$y[k+1] = y[k] + v[k]\sin(\theta[k])\Delta t \tag{2}$$
$$\theta[k+1] = \theta[k] + \frac{v[k]}{L}\tan(\phi[k])\Delta t \tag{3}$$
$$v[k+1] = v[k] + a[k]\Delta t \tag{4}$$

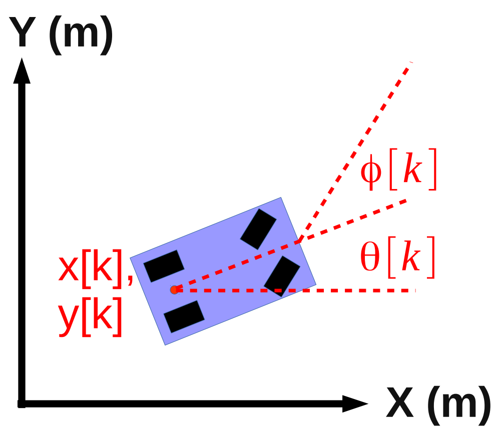


Figure 2: Vehicle Kinematic Model

where

- k, a nonnegative integer, indicates the time step at which we measure the variable (e.g. $v[k]$ is the speed at time step k and $v[k+1]$ is the speed at the following time step)

- $x[k]$ and $y[k]$ denote the coordinates of the vehicle (meters)
- $\theta[k]$ denotes the heading of the vehicle, or the angle with respect to the x-axis (radians)
- $v[k]$ is the speed of the car (meters per second)
- $a[k]$ is the acceleration of the car (meters per second squared)
- $\phi[k]$ is the steering angle input we command (radians)
- Δt is a constant measuring the time difference (in seconds) between time steps $k+1$ and k
- L is a constant and is the length of the car (in meters)

For this problem, let L be 1.0 meter and Δt be 0.1 seconds.

The variables $x[k], y[k], \theta[k], v[k]$ describe the **state** of the car at time step k. The state captures all the information needed to fully determine the current position, speed, and heading of the car. The **inputs** at time step k are $a[k]$ and $\phi[k]$. These are provided by the driver. The current value of these inputs, along with the current state of the vehicle, will determine the state of the vehicle at the next time step.

We note that the problem is nonlinear, due to the sine, cosine and tangent functions, as well as terms including the product of states and inputs.

The purpose of this problem is to show that we can approximate a nonlinear model with a simple linear model and do reasonably well. This is why, despite many systems being nonlinear, linear algebra tools are widely used in practice.

For Parts (b) - (d), fill out the corresponding sections in prob2.ipynb.

(a) We assume that the car has a small heading ($\theta \approx 0$) and that the steering angle is also small ($\phi \approx 0$), where $\approx$ means "approximately equal to." In this case, we could use the following approximations:

$$\begin{aligned} \sin(\alpha) &\approx 0, \\ \cos(\alpha) &\approx 1, \\ \tan(\alpha) &\approx 0. \end{aligned}$$

where α is the small angle of interest. Here, we use a very simple approximation for small angles; in later classes, you may learn better approximations.

Draw, by hand, graphs of of $\sin(\alpha)$ and $\cos(\alpha)$, for α ranging from $-\pi$ to π. Using these graphs can you justify the approximation we are making for small values of α?

(b) Applying the approximation described in the previous part, write down a linear system that approximates the nonlinear vehicle model given above in Equations (1) to (4). In particular, find the 4 x 4 matrix $\mathbf{A}$ and 4 x 2 matrix $\mathbf{B}$ that satisfy the equation given below.

$$\begin{bmatrix} x[k+1] \\ y[k+1] \\ \theta[k+1] \\ v[k+1] \end{bmatrix} = \mathbf{A} \begin{bmatrix} x[k] \\ y[k] \\ \theta[k] \\ v[k] \end{bmatrix} + \mathbf{B} \begin{bmatrix} a[k] \\ \phi[k] \end{bmatrix}$$

Hint: *Start with simplifying Equations* (1) *to* (4).

(c) Suppose we drive the car so that the direction of travel is aligned with the x-axis, and we are driving nearly straight, i.e. the steering angle is $\phi[k] = 0.0001$ radians. (Driving exactly straight would have the steering angle $\phi[k] = 0$ radians.) The initial state and input are:

$$\begin{bmatrix} x[0] \\ y[0] \\ \theta[0] \\ v[0] \end{bmatrix} = \begin{bmatrix} 5.0 \\ 10.0 \\ 0.0 \\ 2.0 \end{bmatrix}$$
$$\begin{bmatrix} a[k] \\ \phi[k] \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.0001 \end{bmatrix}$$

You can use these values in the IPython notebook to compare how the nonlinear system evolves in comparison to the linear approximation that you made. The IPython notebook simulates the car for ten time steps. Are the trajectories similar or very different? Why?

(d) Now suppose we drive the vehicle from the same starting state, but we turn left instead of going straight, i.e. the steering angle is $\phi[k] = 0.5$ radians. The initial state and input are:

$$\begin{bmatrix} x[0] \\ y[0] \\ \theta[0] \\ v[0] \end{bmatrix} = \begin{bmatrix} 5.0 \\ 10.0 \\ 0.0 \\ 2.0 \end{bmatrix}$$
$$\begin{bmatrix} a[k] \\ \phi[k] \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.5 \end{bmatrix}$$

You can use these values in the IPython notebook to compare how the nonlinear system evolves in comparison to the linear approximation that you made. The IPython notebook simulates the car for ten time steps. Are the trajectories similar or very different? Why?

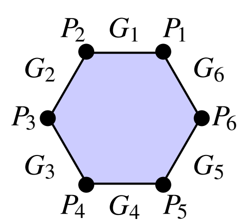
4. Figuring Out The Tips

Learning Objective: *This problem showcases how you can understand general systems of equations by looking at simpler examples. In particular, see if you can generalize your intuition from the case of 5 and 6 guests to a general number of guests.*

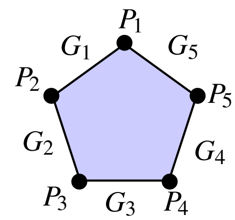
A number of guests gather around a round table for a dinner. Between every adjacent pair of guests, there is a plate for tips. When everyone has finished eating, each person places half their tip in the plate to their left and half in the plate to their right. Suppose you can only see the amount of tips in each plate after everyone has left. Can you deduce the amount that each individual tipped?

Note: For this question, if we assume that tips are positive, then we need to introduce additional constraint that would make the system of equations no longer linear. We are going to ignore this constraint and assume that negative tips are acceptable.

(a) Suppose six guests sit around a table and there are six plates of tips. If we know the amount of tip in each plate, P_1 to P_6, can we determine each individual's tip amount, G_1 to G_6? If yes, explain why by examining the relationship between the plate values, P_1 to P_6, and guest tips, G_1 to G_6. If not, give two different assignments of G_1 to G_6 that will result in the same P_1 to P_6.



(b) Now lets consider five guests at the table, G_1 to G_5, and we can see the amount of tips in the five plates, P_1 to P_5. In this new setting can you figure out each guests tip values, G_1 to G_5?



(c) If n is the total number of guests sitting around a table, for which values of n can you figure out everyone's tip? You do not have to rigorously prove your answer. (**Hint:** consider what is different about parts a and b.)

5. Proof: Linear Dependence in a Square Matrix

(a) Let **A** be a $n \times n$ matrix, (i.e. both the columns and rows are vectors in R^n). Suppose we are told that the columns of **A** are linearly dependent. Prove, then, that the rows of **A** must also be linearly dependent. (**Hint**: A set of vectors $\{\vec{a}_1,...,\vec{a}_n\}$ is linearly dependent if there exist scalars α_1,...,α_n such that $\alpha_1\vec{a}_1$+...+$\alpha_n\vec{a}_n$=$\vec{0}$ and not all α_iâĂŹs are equal to zero.)

(b) Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$$

In other words, we can replace one vector with the sum of itself and another vector and not change their span.

6. Image Stitching

Learning Objective: *This problem is similar to one that students might experience in an upper division computer vision course. Our goal is to give students a flavor of the power of tools from fundamental linear algebra and their wide range of applications.*

Often, when people take pictures of a large object, they are constrained by the field of vision of the camera. This means that they have two options to capture the entire object:

- Stand as far away as they need to include the entire object in the camera's field of view (clearly, we do not want to do this as it reduces the amount of detail in the image)
- (This is more exciting) Take several pictures of different parts of the object and stitch them together like a jigsaw puzzle.

We are going to explore the second option in this problem. Daniel, who is a professional photographer, wants to construct an image by using "image stitching". Unfortunately, Daniel took some of the pictures from different angles as well as from different positions and distances from the object. While processing these pictures, Daniel lost information about the positions and orientations from which the pictures were taken. Luckily, you and your friend Marcela, with your wealth of newly acquired knowledge about vectors and matrices, can help him!

You and Marcela are designing an iPhone app that stitches photographs together into one larger image. Marcela has already written an algorithm that finds common points in overlapping images. **It's your job to**

figure out how to stitch the images together using Marcela's common points to reconstruct the larger image.

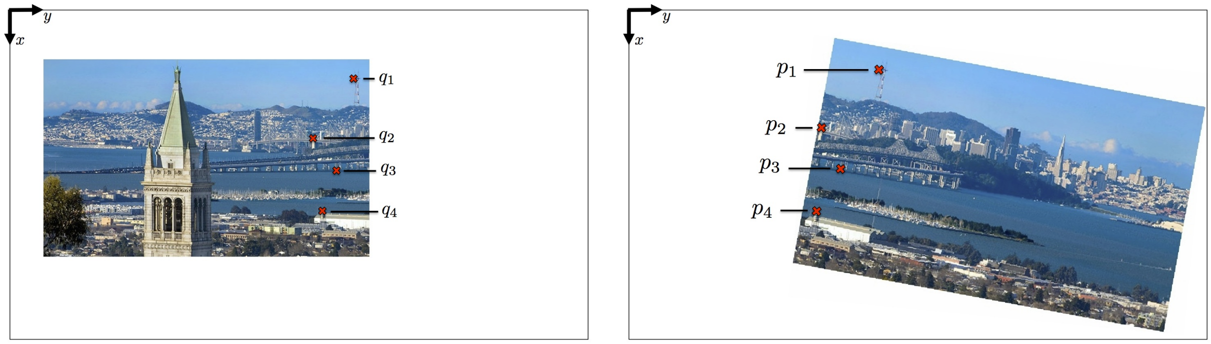


Figure 3: Two images to be stitched together with pairs of matching points labeled.

We will use vectors to represent the common points which are related by a linear transformation. Your idea is to find this linear transformation. For this you will use a single matrix, $\mathbf{R}$, and a vector, $\vec{T}$, that transforms every common point in one image to their corresponding point in the other image. Once you find $\mathbf{R}$ and $\vec{T}$ you will be able to transform one image so that it lines up with the other image.

Suppose $\vec{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$ is a point in one image and $\vec{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$ is the corresponding point in the other image (i.e., they represent the same object in the scene). You write down the following relationship between $\vec{p}$ and $\vec{q}$.

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{R}_{xx} & \mathbf{R}_{xy} \\ \mathbf{R}_{yx} & \mathbf{R}_{yy} \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \underbrace{\begin{bmatrix} T_x \\ T_y \end{bmatrix}}_{\vec{T}} \tag{5}$$

This problem focuses on finding the unknowns (i.e. the components of $\mathbf{R}$ and $\vec{T}$), so that you will be able to stitch the image together.

(a) To understand how the matrix $\mathbf{R}$ and vector $\vec{T}$ transform a vector, $\vec{v}_0$, consider this similar equation,

$$\vec{v}_2 = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \vec{v}_0 + \vec{v}_1. \tag{6}$$

Using $\vec{v}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, what is $\vec{v}_2$? On a single plot, draw the vectors $\vec{v}_0, \vec{v}_1, \vec{v}_2$ in two dimensions. Describe how $\vec{v}_2$ is transformed from $\vec{v}_0$ (e.g. rotated, scaled, shifted).

(b) Multiply Equation (5) out into two scalar linear equations. What are the known values and what are the unknowns in each equation? How many unknowns are there? How many independent equations do you need to solve for all the unknowns? How many pairs of common points $\vec{p}$ and $\vec{q}$ will you need in order to write down a system of equations that you can use to solve for the unknowns?

(c) What is the vector of unknown values? Write out a system of linear equations that you can use to solve for the unknown values (you should use multiple pairs of points $\vec{p}$'s and $\vec{q}$'s to have enough equations based on what you found in part b). Transform these linear equations into a matrix equation, so that we can solve for the vector of unknown values.

(d) In the IPython notebook `prob2.ipynb`, you will have a chance to test out your solution. Plug in the values that you are given for p_x, p_y, q_x, and q_y for each pair of points into your system of equations to solve for the matrix, $\mathbf{R}$, and vector, $\vec{T}$. The notebook will solve the system of equations, apply your transformation to the second image, and show you if your stitching algorithm works. You are not responsible for understanding the image stitching code or Marcela's algorithm.

7. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?