This homework is due July 8th, 2022, at 23:59.
Selfgrades are due July 11th, 2022, at 23:59.

Submission Format
Your homework submission should consist of a single PDF file that contains all of your answers (any hand-written answers should be scanned) as well as your IPython notebook saved as a PDF.

If you do not attach a PDF “printout” of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

1. Reading Assignment [UNGRADED]
For this homework, please read Notes 3, 4, 5, 6 and 7. They will provide an overview of Span, Linear Independence, Introduction to Proofs, State Transition systems, Matrix Inversion and Vector Spaces. You are always welcome and encouraged to read beyond this as well.

2. Span Proofs
Learning Objectives: This is an opportunity to practice your proof development skills. Refer to problem 3 in Discussion 2B for a similar proof with span.

(a) Given some set of vectors \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \} \), show the following:
\[
\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}
\]
In other words, we can replace one vector with the sum of itself and another vector and not change their span.

In order to show this, you have to prove the two following statements:
• If a vector \( \vec{q} \) belongs in \( \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \), then it must also belong in \( \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\} \).
• If a vector \( \vec{r} \) belongs in \( \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\} \), then it must also belong in \( \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \).

In summary, you have to prove the problem statement from both directions.

(b) Consider the span of the set \( \{\vec{v}_1, \ldots, \vec{v}_n, \vec{u}\} \). Suppose \( \vec{u} \) is in the span of \( \{\vec{v}_1, \ldots, \vec{v}_n\} \). Then, show that any vector \( \vec{r} \) in \( \text{span}\{\vec{v}_1, \ldots, \vec{v}_n, \vec{u}\} \) is in \( \text{span}\{\vec{v}_1, \ldots, \vec{v}_n\} \).

3. Basic Circuit Components
Learning Objectives: Review basics of circuit components and current-voltage relationships

(a) Fill in the units for the following quantities:

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Voltage</td>
<td>V</td>
<td></td>
</tr>
<tr>
<td>Current</td>
<td>I</td>
<td></td>
</tr>
<tr>
<td>Resistance</td>
<td>R</td>
<td></td>
</tr>
</tbody>
</table>
(b) What is the voltage across a short circuit (wire)? What is the current going through it? Draw the symbol and sketch the IV relationship.

(c) What is the voltage across an open circuit? What is the current going through it? Draw the symbol and sketch the IV relationship.

(d) What is the relationship between voltage and current for a resistor? Draw the symbol and IV relationship.

(e) What is the voltage across a voltage source $V_s$? What is the current going through it? Draw the symbol and sketch the IV relationship.

(f) What is the voltage across a current source $I_s$? What is the current going through it? Draw the symbol and sketch the IV relationship.

4. Ohm’s Law

**Learning Objectives:** Practice implementing Ohm’s Law in basic circuits.

(a) Take the circuit diagram below.

$V_s = 5$ V and $R = 10$ Ω.

i. How many nodes are in this circuit?

ii. What is the potential at ground?

iii. What is $V_R$, the voltage across $R$?

iv. What is $I_R$, the current through $R$?

(b) Now switch the voltage source with a current source.

$I_s = 0.002$ A = 2 mA and $R = 10,000,000$ Ω = 10 MΩ.

i. How many nodes are in this circuit?
ii. What is the potential at ground?
iii. What is \( I_R \), the current through \( R \)?
iv. What is \( V_R \), the voltage across \( R \)?

(c) Now short the circuit.

i. How many nodes are in this circuit?
ii. What is \( V_R \), the voltage across \( R \)?
iii. What is \( I_R \), the current through \( R \)?

5. Linear Dependence in a Square Matrix

Learning Objective: This is an opportunity to practice applying proof techniques. This question is specifically focused on linear dependence of rows and columns in a square matrix.

Let \( A \) be a square \( n \times n \) matrix, (i.e. both the columns and rows are vectors in \( \mathbb{R}^n \)). Suppose we are told that the columns of \( A \) are linearly dependent. Prove, then, that the rows of \( A \) must also be linearly dependent. You can use the following conclusion in your proof:

*If Gaussian elimination is applied to a matrix \( A \), and the resulting matrix (in reduced row echelon form) has at least one row of all zeros, this means that the rows of \( A \) are linearly dependent.*

(Hint: Can you use the linear dependence of the columns to say something about the number of solutions to \( A\mathbf{x} = \mathbf{0} \)? How does the number of solutions relate to the result of Gaussian elimination?)

6. Social Media

Learning Objective: Practice setting up transition matrices from a diagram and understand how to compute subsequent states of the system.

As a tech-savvy Berkeley student, the distractions of streaming services are always calling you away from productive stuff like homework for your classes. You’re curious—are you the only one who spends hours switching between Netflix or Hulu? How do other students manage to get stuff done and balance staying up to date with the Bachelor? You conduct an experiment, collect some data, and notice Berkeley students tend to follow a pattern of behavior similar to the figure below. So, for example, if \( x = 100 \) students are on Netflix, in the next timestep, \( 0.2x \) (20) of them will click on a link and move to Hulu, and \( 0.8x \) (80) will remain on Netflix.
(a) Let us define $x_N[n]$ as the number of students on Netflix at timestep $n$, $x_H[n]$ as the number of students on Hulu at timestep $n$, $x_C[n]$ as the number of students watching any kind of cat video at timestep $n$, and $x_W[n]$ as the number of students working at timestep $n$. Let the state vector be: $\vec{x}[n] = \begin{bmatrix} x_N[n] \\ x_H[n] \\ x_C[n] \\ x_W[n] \end{bmatrix}$. Derive the corresponding transition matrix.

Hint: A transition matrix, $A$ is the matrix that connects $\vec{x}[n]$, the vector at timestep $n$ to the vector at timestep $n+1$: $\vec{x}[n+1] = A\vec{x}[n]$

(b) There are 1500 of you in the class. Suppose on a given Friday evening (the day when HW is due), there are 700 EECS16A students on Netflix, 450 on Hulu, 200 watching Cat Videos, and 150 actually doing work. In the next timestep, how many people will be doing each activity? In other words, after you apply the matrix once to reach the next timestep, what is the state vector?

(c) Compute the sum of each column in the state transition matrix. What is the interpretation of this?

7. Image Stitching

Learning Objective: This problem is similar to one that students might experience in an upper division computer vision course. Our goal is to give students a flavor of the power of tools from fundamental linear algebra and their wide range of applications.

Often, when people take pictures of a large object, they are constrained by the field of vision of the camera. This means that they have two options to capture the entire object:

- Stand as far away as they need to include the entire object in the camera’s field of view (clearly, we do not want to do this as it reduces the amount of detail in the image)
- (This is more exciting) Take several pictures of different parts of the object and stitch them together like a jigsaw puzzle.

We are going to explore the second option in this problem. Daniel, who is a professional photographer, wants to construct an image by using “image stitching”. Unfortunately, Daniel took some of the pictures from different angles as well as from different positions and distances from the object. While processing these pictures, Daniel lost information about the positions and orientations from which the pictures were taken. Luckily, you and your friend Marcela, with your wealth of newly acquired knowledge about vectors and matrices, can help him!
You and Marcela are designing an iPhone app that stitches photographs together into one larger image. Marcela has already written an algorithm that finds common points in overlapping images. **It’s your job to figure out how to stitch the images together using Marcela’s common points to reconstruct the larger image.**

We will use vectors to represent the common points which are related by a linear transformation. Your idea is to find this linear transformation. For this you will use a single matrix, $\mathbf{R}$, and a vector, $\mathbf{t}$, that transforms every common point in one image to their corresponding point in the other image. Once you find $\mathbf{R}$ and $\mathbf{t}$ you will be able to transform one image so that it lines up with the other image.

Suppose $\mathbf{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$ is a point in one image, which is transformed to $\mathbf{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$ is the corresponding point in the other image (i.e., they represent the same object in the scene). For example, Fig. 1 shows how the points $\mathbf{p}_1, \mathbf{p}_2 ...$ in the right image are transformed to points $\mathbf{q}_1, \mathbf{q}_2 ...$ on the left image. You write down the following relationship between $\mathbf{p}$ and $\mathbf{q}$.

$$
\begin{bmatrix}
q_x \\
q_y
\end{bmatrix} =
\begin{bmatrix}
r_{xx} & r_{xy} \\
r_{yx} & r_{yy}
\end{bmatrix}
\begin{bmatrix}
p_x \\
p_y
\end{bmatrix} +
\begin{bmatrix}
r_x \\
r_y
\end{bmatrix}
\mathbf{t}
$$

This problem focuses on finding the unknowns (i.e. the components of $\mathbf{R}$ and $\mathbf{t}$), so that you will be able to stitch the image together.

(a) To understand how the matrix $\mathbf{R}$ and vector $\mathbf{t}$ transforms any vector representing a point on a image, consider this equation similar to Equation (1),

$$
\mathbf{v} = \begin{bmatrix}
2 \\
-2
\end{bmatrix} \mathbf{u} + \mathbf{w} = \mathbf{v}_1 + \mathbf{w}.
$$

Use $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for this part.

We want to find out what geometric transformation(s) can be applied on $\mathbf{u}$ to give $\mathbf{v}$.

**Step 1:** Find out how $\begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$ is transforming $\mathbf{u}$. Evaluate $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \mathbf{u}$.  

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What **geometric transformation(s)** might be applied to \( \vec{u} \) to get \( \vec{v}_1 \)? Choose the options that answers the question and explain your choice.

(i) Rotation  
(ii) Scaling  
(iii) Shifting/Translation

*Drawing the vectors \( \vec{u}, \vec{v}_1 \) in two dimensions on a single plot might help you to visualize the transformations. You can also look into the corresponding demo in the IPython notebook prob4.ipynb.*

**Step 2:** Find out \( \vec{v}_1 = \vec{v}_1 + \vec{w} \). Find out how addition of \( \vec{w} \) is geometrically transforming \( \vec{v}_1 \). Choose the option that answers the question and explain your choice.

(i) Rotation  
(ii) Scaling  
(iii) Shifting/Translation

*Drawing the vectors \( \vec{v}, \vec{w}, \vec{v}_1 \) in two dimensions on a single plot might help you to visualize the transformations. You can also look into the corresponding demo in the IPython notebook prob4.ipynb.*

(b) **Multiply Equation (1) out into two linear equations.**

(i) What are the known values and what are the unknown values in each equation? 
(ii) How many unknown values are there? 
(iii) How many independent equations do you need to solve for all the unknowns? 
(iv) How many pairs of common points \( \vec{p} \) and \( \vec{q} \) will you need in order to write down a system of equations that you can use to solve for the unknowns? *Hint: Remember that each pair of \( \vec{p} \) and \( \vec{q} \) will give you two different linear equations.*

(c) Use what you learned in the above two subparts to explicitly write out **just enough** linear equations of these transformations as you need to solve the system. Assume that the four pairs of points from Fig. 1 are labeled as:

\[
\vec{q}_1 = \begin{bmatrix} q_{1x} \\ q_{1y} \end{bmatrix}, \quad \vec{p}_1 = \begin{bmatrix} p_{1x} \\ p_{1y} \end{bmatrix}, \quad \vec{q}_2 = \begin{bmatrix} q_{2x} \\ q_{2y} \end{bmatrix}, \quad \vec{p}_2 = \begin{bmatrix} p_{2x} \\ p_{2y} \end{bmatrix}, \quad \vec{q}_3 = \begin{bmatrix} q_{3x} \\ q_{3y} \end{bmatrix}, \quad \vec{p}_3 = \begin{bmatrix} p_{3x} \\ p_{3y} \end{bmatrix}, \quad \vec{q}_4 = \begin{bmatrix} q_{4x} \\ q_{4y} \end{bmatrix}, \quad \vec{p}_4 = \begin{bmatrix} p_{4x} \\ p_{4y} \end{bmatrix}.
\]

(d) Remember that we ultimately want to solve for the components of the \( R \) matrix and the vector \( \vec{t} \); so we should try to isolate these. We can represent these unknowns in terms of a vector, \( \vec{\alpha} = \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{yx} \\ r_{yy} \\ t_x \\ t_y \end{bmatrix} \).

Translate your equations from the previous part into a matrix-vector formulation that will allow you to solve for \( \vec{\alpha} \).

(e) In the IPython notebook prob4.ipynb, you will have a chance to test out your solution. Plug in the values that you are given for \( p_x, p_y, q_x, \) and \( q_y \) for each pair of points into your system of equations to solve for the matrix, \( R \), and vector, \( \vec{t} \). The notebook will solve the system of equations, apply your transformation to the second image, and show you if your stitching algorithm works. **You are NOT responsible for understanding the image stitching code or Marcela’s algorithm.**
8. Mechanical Inverses

Learning Objectives: Matrices represent linear transformations, and their inverses represent the opposite transformation. Here we practice inversion, but are also looking to develop an intuition. Visualizing the transformations might help develop this intuition.

For each of the following values of matrix A:

i. Find the inverse, $A^{-1}$, if it exists. If you find that the inverse does not exist, mention how you decided that. Solve this by hand.

ii. For parts (a)-(d), in addition to finding the inverse (if it exists), describe how the matrix $A$ transforms an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

For example, if $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$, then $A$ could scale $\begin{bmatrix} x \\ y \end{bmatrix}$ by 2 to get $\begin{bmatrix} 2x \\ 2y \end{bmatrix}$. If $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$, then $A$ could reflect $\begin{bmatrix} x \\ y \end{bmatrix}$ across the $x$ axis, etc. *Hint: It may help to plot a few examples to recognize the pattern.*

iii. For parts (a)-(d), if we use $A$ to geometrically transform $\begin{bmatrix} x \\ y \end{bmatrix}$ to get $\begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$, is it possible to reverse the transformation geometrically, i.e. is it possible to retrieve $\begin{bmatrix} x \\ y \end{bmatrix}$ from $\begin{bmatrix} u \\ v \end{bmatrix}$ geometrically?

(a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

Assume $\cos \theta \neq 0$. *Hint: $\cos^2 \theta + \sin^2 \theta = 1$.*

(e) $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$

(f) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 4 & 4 \end{bmatrix}$

(g) (Optional) $A = \begin{bmatrix} -1 & 1 & -\frac{1}{2} \\ 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}$

(h) (Optional) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

(i) (Optional) $A = \begin{bmatrix} 3 & 0 & -2 & 1 \\ 0 & 2 & -1 & 3 \\ 3 & 1 & 0 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix}$
9. Matrix Proofs

(a) Show that for general matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$:

$$(AB)^T = B^T A^T$$

**Hint:** For two vectors, $\vec{a}$ and $\vec{b}$, $\vec{a}^T \vec{b} = \vec{b}^T \vec{a}$

(b) (OPTIONAL) Show that for general matrices $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times p}$, and $C \in \mathbb{R}^{p \times q}$

$$(AB)C = A(BC)$$

10. Subspaces, Bases and Dimension

For each of the sets $U$ (which are subsets of $\mathbb{R}^3$) defined below, state whether $U$ is a subspace of $\mathbb{R}^3$ or not. If $U$ is a subspace, find a basis for it and state the dimension. You have to show that all three properties of a subspace (as mentioned in Note 8) hold.

(a) $U = \left\{ \begin{bmatrix} 2(x+y) \\ x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

(b) $U = \left\{ \begin{bmatrix} x \\ y \\ z+1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$

(c) $U = \left\{ \begin{bmatrix} x \\ y \\ x+1 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

(d) $U = \left\{ \begin{bmatrix} x \\ y \\ x+y^2 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

11. Homework Process and Study Group

Who did you work with on this homework? List names and student ID’s. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.