Eigen-Stuff, Convergence, and Steady State

Remember the definition of eigenvectors and eigenvalues:

\[ A \vec{v} = \lambda \vec{v} \]

where \( \vec{v} \) is an eigenvector and \( \lambda \) is an eigenvalue. For the purposes of steady state/convergence analysis, we think of \( A \) as an operator which acts on some state vector \( \vec{v} \) at each time step. All behavior can be generalized down to a few cases:

- \( \lambda = 1 \): Any vector in the eigenspace associated with \( \lambda = 1 \) \( \vec{v}_\lambda=1 \) is a “steady state”. In other words,
  \[ A \vec{v}_\lambda=1 = \vec{v}_\lambda=1 \]
  So if we keep applying \( A \) to \( \vec{v}_\lambda=1 \), the state doesn’t change.

- \( \lambda = -1 \): Say it has an eigenspace containing \( \vec{v}_\lambda=-1 \). Let’s call some vector in that eigenspace \( \vec{v}_\lambda=-1 \).
  If we keep applying \( A \) to this vector, it just bounces back and forth between the positive and negative versions of the vector:
  \[ \lim_{n \to \infty} (A^n \vec{v}_\lambda=-1) = \lim_{n \to \infty} (-1)^n \vec{v}_\lambda=-1 \]
  The limit doesn’t exist! It just goes back and forth forever and ever, meaning if you start out the system in state \( \vec{v}_\lambda=-1 \), it will never converge. This shows up in controls when people talk about “marginal instability” and in circuits when people talk about “ringing”.

- \( |\lambda| < 1 \): We’ll say this eigenvalue has an eigenspace containing \( \vec{v}_{<1} \). Repeating the process from above:
  \[ \lim_{n \to \infty} (A^n \vec{v}_{<1}) = \lim_{n \to \infty} \lambda^n \vec{v}_{<1} = \vec{0} \]
  Because \( |\lambda| < 1 \), multiplying it by itself repeatedly just makes the magnitude smaller and smaller. This holds whether \( \lambda \) is positive or negative, and what this means is that if the system starts out in \( \vec{v}_{<1} \), after an infinite amount of time the state will approach \( \vec{0} \). You can think of it as asymptotically approaching zero. For physical systems, (not in scope of the class) this is related to damping (overdamped being \( \lambda > 0 \) and underdamped being \( \lambda < 0 \)).

- \( |\lambda| > 1 \): Now we’ll say this eigenvalue has an eigenspace containing \( \vec{v}_{>1} \).
  \[ \lim_{n \to \infty} (A^n \vec{v}_{>1}) = \lim_{n \to \infty} \lambda^n \vec{v}_{>1} \]
  In this case, the state vector just keeps increasing in size to infinity, meaning if you start out the system in state \( \vec{v}_{>1} \), your system will never converge to a steady state.
The question then arises: what happens if we have a state vector \( \vec{s} \) which is not an eigenvector? We can think of \( \vec{s} \) as a linear combination of the \( N \) eigenvectors:

\[
A \vec{v}_i = \lambda_i \vec{v}_i
\]

\[
\vec{s} = \sum_{i=1}^{N} c_i \vec{v}_i
\]

After many many time steps:

\[
\lim_{n \to \infty} (A^n \vec{s}) = \lim_{n \to \infty} \left( \sum_{i=1}^{N} c_i \lambda_i^n \vec{v}_i \right)
\]

Now what we care about is the coefficients \( c_i \) as it applies to the cases we described above. We’ve now broken our state \( \vec{s} \) into its eigenvector components, and we know how those components will behave when acted upon by \( A \) and thus, what \( \vec{s} \) will look like after applying \( A \) many times.

1. A Tropical Tale of Triumph: Does Pineapple Come Out on Top? (Fall 2018 Midterm 1)

(Based on a true story) During a discussion section, one of your TAs, Nick, makes the claim that pineapple belongs on pizza. Another TA, Elena, strongly disagrees. Naturally, a war starts and students begin to flock to the TA they agree with, switching discussion sections every week. Some students don’t have an opinion and go to Lydia’s section since she is neutral in the matter. As a 16A student, you want to analyze this war to see how it will play out.

(a) You manage to capture the behavior of the students as a transition matrix, but want to visualize it.

You’ve written out the transition matrix \( M \):

\[
M = \begin{bmatrix}
0.5 & 0 & 0 \\
0.25 & 0.5 & 1 \\
0.25 & 0.5 & 0
\end{bmatrix}
\]

such that

\[
\begin{bmatrix}
x_{Elena}[n+1] \\
x_{Nick}[n+1] \\
x_{Lydia}[n+1]
\end{bmatrix} = M \begin{bmatrix}
x_{Elena}[n] \\
x_{Nick}[n] \\
x_{Lydia}[n]
\end{bmatrix}.
\]

Each element of the state vector \( \vec{x}[n] = [x_{Elena}[n] \ x_{Nick}[n] \ x_{Lydia}[n]]^T \) represents the number of students attending that section at timestep \( n \). Fill in values in the boxes in Figure 1 below such that the diagram represents the transition matrix \( M \).
(b) Your friend Vlad tells you that your transition matrix $M$ was wrong, and gives you a new transition matrix $S$, which has a steady state. In order to find who wins the war, you need to find how many students end up in each section after everything has settled. **Find a vector $\vec{x}$ that represents a steady state of $S$.**

$$S = \begin{bmatrix} 0.2 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0.3 & 0 & 1 \end{bmatrix}$$

(c) Your other friend Gireeja points out that the arguments are causing new people to join the sections and others to leave entirely. In other words, the system is not conservative! The new system can be...
modeled with a state transition matrix $A$ that has the following eigenvalue/eigenvector pairings:

$$
\lambda_1 = 1 : \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\lambda_2 = \frac{1}{2} : \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
$$

$$
\lambda_3 = 2 : \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
$$

You want the number of students in sections to stabilize. Which of the vectors below represent steady states of the system, i.e. $\vec{x}$ such that $A\vec{x} = \vec{x}$? Fill in the circle(s) to the left of these vector(s).

$$
\begin{array}{cccc}
\circ & \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} & \circ & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \circ & \begin{bmatrix} 513 \\ 513 \\ 0 \end{bmatrix} & \circ & \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix} \\
\circ & \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & \circ & \begin{bmatrix} 1026 \\ 0 \\ 0 \end{bmatrix} & \circ & \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} & \circ & \begin{bmatrix} 0 \\ 1026 \\ 0 \end{bmatrix}
\end{array}
$$

(d) Assume we are still working with the same state transition matrix $A$ as in part (c). Which of the vectors below represent initial states such that the number of students in the sections keeps growing? Fill in the circle(s) to the left of these vector(s).

$$
\begin{array}{cccc}
\circ & \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} & \circ & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \circ & \begin{bmatrix} 513 \\ 513 \\ 0 \end{bmatrix} & \circ & \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix} \\
\circ & \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & \circ & \begin{bmatrix} 1026 \\ 0 \\ 0 \end{bmatrix} & \circ & \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} & \circ & \begin{bmatrix} 0 \\ 1026 \\ 0 \end{bmatrix}
\end{array}
$$
(e) Again assume we are still working with the same state transition matrix $A$ as in part (c). Which of the vectors below represent initial states such that everyone leaves the system, i.e. $\lim_{n \to \infty} A^n \vec{x} = \vec{0}$? Fill in the circle(s) to the left of these vector(s).

\[
\begin{array}{llll}
\circ & \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} & \circ & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \circ & \begin{bmatrix} 513 \\ 0 \\ 0 \end{bmatrix} & \circ & \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix} \\
\circ & \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & \circ & \begin{bmatrix} 1026 \\ 0 \\ 0 \end{bmatrix} & \circ & \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} & \circ & \begin{bmatrix} 0 \\ 1026 \\ 0 \end{bmatrix}
\end{array}
\]

(f) Let us generalize the idea of convergence. Consider the following system:

$$\vec{x}_{n+1} = T\vec{x}_n$$

where $\vec{x}$ is a vector with $N$ elements and $T$ is any $N \times N$ matrix unrelated to the previous parts. $T$ has $N$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$, and $N$ associated eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N$ such that $T\vec{v}_i = \lambda_i \vec{v}_i$ for $1 \leq i \leq N$. Let $|\lambda_1| > 1$. Prove that there exists at least one initial state $\vec{x}_0$ for this system such that it does not converge to a steady state.