1. Mechanical Inner Products

For the following pairs of vectors, find the Euclidean inner product $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y}$.

(a) \[
\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}, \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
1 \\
0 \\
3
\end{bmatrix}, \begin{bmatrix}
-3 \\
2 \\
1
\end{bmatrix}
\]

2. Geometric Interpretation of the Inner Product

In this problem, we explore the geometric interpretation of the Euclidean inner product, restricting ourselves to vectors in $\mathbb{R}^2$.

Remember that the formula for the inner product of two vectors can be expressed in terms of their magnitudes and the angle between them as follows:

$$\langle \vec{x}, \vec{y} \rangle = ||\vec{x}|| ||\vec{y}|| \cos \theta$$

The figure below may be helpful in illustrating this property:

![Geometric Interpretation Diagram]
For each sub-part, give an example of any two (nonzero) vectors \( \vec{x}, \vec{y} \in \mathbb{R}^2 \) that satisfy the stated condition and compute their inner product.

(a) Give an example of a pair of parallel vectors (vectors that point in the same direction and have an angle of 0 degrees between them).

(b) Give an example of a pair of anti-parallel vectors (vectors that point in opposite directions).

(c) Give an example of a pair of perpendicular vectors (vectors that have an angle of 90 degrees between them).

3. From Inner Products To Projections

Given that \( \langle \vec{x}, \vec{y} \rangle \) is a measure of similarity between two vectors, let’s try to use this to find how much of one vector \( \vec{y} \) is in the direction of another vector \( \vec{x} \).

(a) Let’s start with \( \langle \vec{x}, \vec{y} \rangle \). We want a quantity that is independent of the norm of \( \vec{x} \), \( ||\vec{x}|| \). Is \( \langle \vec{x}, \vec{y} \rangle \) independent of the norm? Consider \( \langle \vec{x}, \vec{y} \rangle \) for the examples below.

\[
\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]

(b) Suppose we divide \( \langle \vec{x}, \vec{y} \rangle \) by the norm of \( \vec{x} \), \( ||\vec{x}|| \), to get \( \frac{\langle \vec{x}, \vec{y} \rangle}{||\vec{x}||} \). Is this new quantity independent of the norm of \( \vec{x} \)? Test it on the examples above.

(c) We now have a scalar quantity that represents how much of \( \vec{y} \) is in the direction of \( \vec{x} \). Now let’s look for a vector \( \vec{z} \) that has a norm of \( \frac{\langle \vec{x}, \vec{y} \rangle}{||\vec{x}||} \) and points in the same direction as \( \vec{x} \). The vector \( \vec{z} \) should be written in terms of \( \vec{x} \) and \( \vec{y} \).

(d) Consider the quantity \( \frac{\langle \vec{x}, \vec{y} \rangle}{||\vec{x}|| ||\vec{y}||} \). What is the maximum this quantity could be? When does this occur? What is the minimum this quantity could be? When does this occur?

Hint: Use the Cauchy-Schwartz inequality.

(e) We define the angle between two vectors as \( \cos(\theta) = \frac{\langle \vec{x}, \vec{y} \rangle}{||\vec{x}|| ||\vec{y}||} \). When do two vectors have an angle of 90° between them? When do they have an angle of 0°? When do they have an angle of 180°?