1. Mechanical Determinants

(a) Compute the determinant of \[
\begin{pmatrix}
2 & 0 \\
0 & 3 \\
\end{pmatrix}
\].

(b) Compute the determinant of \[
\begin{pmatrix}
2 & -3 & 1 \\
2 & 0 & -1 \\
1 & 4 & 5 \\
\end{pmatrix}
\].

(c) We know that the determinant of a matrix represents the multi-dimensional volume formed by the column vectors. Explain geometrically why the determinant of a matrix with linearly dependent column vectors is always 0.
2. Mechanical Eigenvalues and Eigenvectors

In each part, find the eigenvalues of the matrix $M$ and the associated eigenvectors.

(a) $M = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

Do you observe anything?

(b) $M = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

(c) Without calculation, determine whether the identity matrix in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$. What are the corresponding eigenvectors?
3. Eigenvalues and Special Matrices – Visualization

An eigenvector $\vec{v}$ belonging to a square matrix $A$ is a nonzero vector that satisfies

$$A\vec{v} = \lambda\vec{v}$$

where $\lambda$ is a scalar known as the eigenvalue corresponding to eigenvector $\vec{v}$. Rather than mechanically compute the eigenvalues and eigenvectors, answer each part here by reasoning about the matrix at hand.

(a) Does the identity matrix in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

(b) Does a diagonal matrix

$$\begin{bmatrix}
d_1 & 0 & 0 & \cdots & 0 \\
0 & d_2 & 0 & \cdots & 0 \\
0 & 0 & d_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & d_n
\end{bmatrix}$$

in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

(c) Conceptually, does a rotation matrix in $\mathbb{R}^2$ by angle $\theta$ have any eigenvalues $\lambda \in \mathbb{R}$? For which angles is this the case?
(d) **(PRACTICE)** Now let us mechanically compute the eigenvalues of the rotation matrix in $\mathbb{R}^2$. Does it agree with our findings above? As a refresher, the rotation matrix $\mathbf{R}$ has the following form:

$$
\mathbf{R} = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix}
$$

(e) Does the reflection matrix $\mathbf{T}$ across the x-axis in $\mathbb{R}^{2\times2}$ have any eigenvalues $\lambda \in \mathbb{R}$?

$$
\mathbf{T} = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
$$

(f) If a matrix $\mathbf{M}$ has an eigenvalue $\lambda = 0$, what does this say about its null space? What does this say about the solutions of the system of linear equations $\mathbf{M}\mathbf{x} = \mathbf{b}$?
(g) (Practice) Does the matrix \[
\begin{bmatrix}
1 & 1 \\
0 & 0 \\
\end{bmatrix}
\] have any eigenvalues \( \lambda \in \mathbb{R} \)? What are the corresponding eigenvectors?