1. Visualizing Span

We are given a point \( \vec{c} \) that we want to get to, but we can only move in two directions: \( \vec{a} \) and \( \vec{b} \). We know that to get to \( \vec{c} \), we can travel along \( \vec{a} \) for some amount \( \alpha \), then change direction, and travel along \( \vec{b} \) for some amount \( \beta \). We want to find these two scalars \( \alpha \) and \( \beta \), such that we reach point \( \vec{c} \). That is, \( \alpha \vec{a} + \beta \vec{b} = \vec{c} \).

\[
\begin{align*}
\vec{c} & \quad \vec{b} \\
\vec{a} & \\
\end{align*}
\]

(a) Formulate the system of equations as a matrix to find the unknowns, \( \alpha, \beta \), in terms of the vectors \( \vec{a}, \vec{b}, \vec{c} \).

(b) First, consider the case where \( \vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), and \( \vec{c} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \). Draw these vectors on a sheet of paper. Now find the two scalars \( \alpha \) and \( \beta \), such that we reach point \( \vec{c} \). What are these scalars if we use \( \vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( \vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \) instead?

2. Span Basics

(a) What is span \( \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \)?

(b) Is \( \begin{bmatrix} 5 \\ 5 \\ 0 \\ 0 \end{bmatrix} \) in span \( \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \)?

(c) What is a possible choice for \( \vec{v} \) that would make span \( \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} = \mathbb{R}^3 \)?

(d) For what values of \( b_1, b_2, b_3 \) is the following system of linear equations consistent? (“Consistent” means there is at least one solution.)

\[
\begin{bmatrix}
1 & 2 \\
2 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
\]
3. Span Proofs

Given some set of vectors \( \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \), show the following:

(a) \[
\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\alpha \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}, \text{ where } \alpha \text{ is a non-zero scalar}
\]

In other words, we can scale our spanning vectors and not change their span.

(b) (for practice) \[
\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_2, \vec{v}_1, \ldots, \vec{v}_n\}
\]

In other words, we can swap the order of our spanning vectors and not change their span.