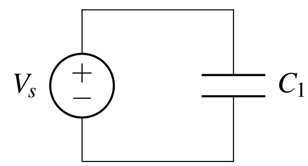
EECS 16A Designing Information Devices and Systems I
Spring 2020 Discussion 8B

1. Voltages Across Capacitors

For the circuits given below, calculate the voltage across the capacitors. For parts (a) and (b) only, also calculate the charge and energy stored in each capacitor. Let $C_1 = 1\,\mu\text{F}$, $C_2 = 3\,\mu\text{F}$, $V_s = 1\,\text{V}$, and $I_s = 2\,\text{mA}$.

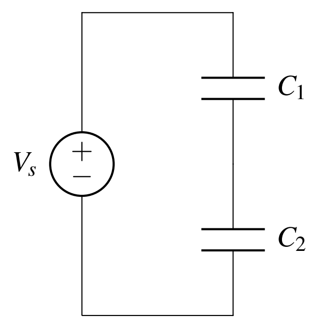
(a)



Answer:

The capacitor is charged by the voltage source, the value of which is V_s. Hence, the voltage across the capacitor has to be V_s. The charge is $q = C_1 V_s = 1\mu\text{C}$ ($+q$ accumulates on the top plate, $-q$ on the bottom plate). The energy stored is $E = \frac{C_1 V_s^2}{2} = \frac{1}{2}\mu\text{J}$.

(b)



Answer:

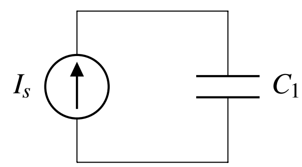
$$C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

The series equivalent capacitance is $C_{eq} = \frac{3}{4}\mu\text{F}$, so $Q_{eq} = C_{eq}V = \frac{3}{4}\mu\text{C}$. Note that this means $+\frac{3}{4}\mu\text{C}$ is on the top plate of C_1 and $-\frac{3}{4}\mu\text{C}$ is on the bottom plate of C_2. Hence, $-\frac{3}{4}\mu\text{C}$ is stored on the bottom plate of C_1 and $+\frac{3}{4}\mu\text{C}$ on the top plate of C_2. This result agrees with the conservation of charge, since the net charge on the middle node node before and after connecting the voltage source remains zero. It also means that the same amount of charge, $Q_{eq} = \frac{3}{4}\mu\text{C}$ is stored on both capacitors.

The voltage V_1 across C_1 is $\frac{Q_{eq}}{C_1} = \frac{3}{4}\text{V}$. The voltage V_2 across C_2 is $\frac{Q_{eq}}{C_2} = \frac{1}{4}\text{V}$.

The charge stored on both is Q_{eq} as mentioned above. The energy stored can be calculated as $\frac{1}{2}CV^2$ for each capacitor, so $E_1 = 280\,\text{nJ}$ and $E_2 = 94\,\text{nJ}$.

(c)



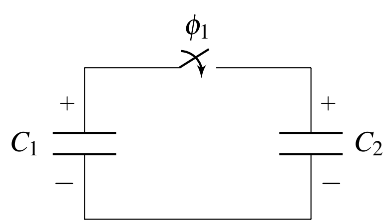
Answer:

$$I_{\mathrm{C1}} = C\frac{dV_{\mathrm{C1}}}{dt}$$
$$I_{\mathrm{C1}} = I_{\mathrm{s}}$$
$$\int dV_{\mathrm{C}} = \int \frac{I_{\mathrm{s}}}{C_1} dt$$
$$V_{\mathrm{C1}}(t) = \frac{I_{\mathrm{s}}}{C_1} t$$
$$V_{\mathrm{C1}}(t) = \frac{2\,\mathrm{mA}}{1\,\mu\mathrm{F}} t$$
$$V_{\mathrm{C1}}(t) = 2000t$$

Using the I-V relation for capacitance, we find that, when a constant current source is applied to a capacitor, the capacitor's voltage will increase linearly as a function of t, with the current and capacitance determining the rate of change.

2. Capacitors and Charge Conservation (with Energy!)

(a) Consider the circuit below with $C_1 = C_2 = 1\,\mu\mathrm{F}$ and an open switch. Suppose that C_1 is initially charged to $+1\,\mathrm{V}$ and that C_2 is charged to $+2\,\mathrm{V}$. How much charge is on C_1 and C_2? How much energy is stored in each of the capacitors? What is the total stored energy?



Answer:

$$q_1 = C_1 V_1 = 1\,\mu\mathrm{C}$$
$$q_2 = C_2 V_2 = 2\,\mu\mathrm{C}$$

Energy:

$$E = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV$$

Note: We include a factor of $\frac{1}{2}$ because when we charge a capacitor, we do it incrementally. Not all the charge Q is moved against a potential of V. We are basically taking $\int v(q)\,dq = \int \frac{q}{C}\,dq = \frac{1}{2}\frac{Q^2}{C}$.
Therefore, $E_1 = \frac{1}{2}C_1V_1^2 = 0.5\,\mu\mathrm{J}$, $E_2 = \frac{1}{2}C_2V_2^2 = 2\,\mu\mathrm{J}$, and the total energy is $2.5\,\mu\mathrm{J}$.
Note: Does it make sense that C_2 has more than twice the energy of C_1 even though it is only twice the voltage? (Yes, C_2 has *more charge* at twice the voltage.)

(b) Now the switch is closed (i.e. the capacitors are connected together.) Is charge conserved? What are the voltages across and the charges on C_1 and C_2? What is the total stored energy?

Answer:

Charge is always conserved.

Let $Q_{C_1,1}, Q_{C_2,1}$ be the charges on the capacitors after the switch is closed. There was $3\,\mu\text{C}$ of total charge on the top two plates of the capacitors initially, so we must have

$$Q_{C_1,1} + Q_{C_2,1} = 3\,\mu\text{C}$$

Further, since the capacitors are connected in parallel the voltages across them must be the same, so:

$$\frac{Q_{C_1,1}}{C_1} = \frac{Q_{C_2,1}}{C_2}$$

Solving this system gives:

$$Q_{C_1,1} = Q_{C_2,1} = 1.5\,\mu\text{C}$$

Comparing to the previous part, charge has moved from C_2 to C_1. This yields a voltage of $1.5\,\text{V}$, using $V = \frac{Q_{C_1,1}}{C_1} = \frac{Q_{C_2,1}}{C_2}$.

The energies are:

$$E_1 = \frac{1}{2}(1.5\,\mu\text{C})(1.5\,\text{V}) = 1.125\,\mu\text{J}$$

$$E_2 = \frac{1}{2}(1.5\,\mu\text{C})(1.5\,\text{V}) = 1.125\,\mu\text{J}$$

Total energy: $E = 2.25\,\mu\text{J}$

(c) Is there more or less energy than before the switch was closed? Why?

Answer:

The energy decreases when the switch is closed. First, the energy could not possibly be more since there is no energy input to the system. But there's no energy output either – where did the energy go? The wires have an internal resistance that would dissipate this energy.

(d) Answer the above three questions but now with $C_1 = 2\,\mu\text{F}$ and $C_2 = 1\,\mu\text{F}$. Suppose that they are initially charged in the same way: C_1 is charged to $+1\,\text{V}$, and C_2 is charged to $+2\,\text{V}$.

Answer:

Now, the initial charges on the capacitors will be the same. However, after the switch closes the voltages across the caps need to be the same (parallel connection) so some amount of charge will flow from the smaller cap to to the larger one in order to satisfy

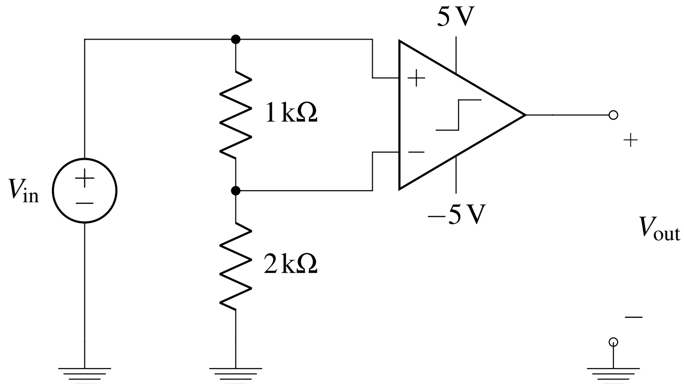
$$V_C = \frac{Q_{C_1,1}}{C_1} = \frac{Q_{C_2,1}}{C_2}$$

.

3. Comparators

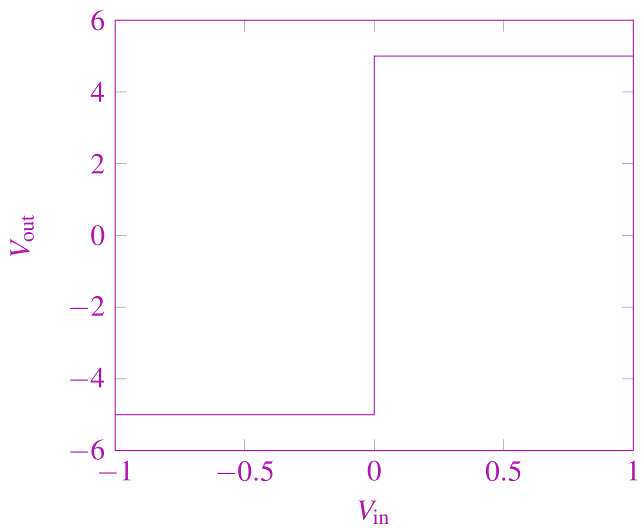
For each of the circuits shown below, plot V_{out} for V_{in} ranging from $-10\,\text{V}$ to $10\,\text{V}$ for part (a) and from $0\,\text{V}$ to $10\,\text{V}$ for part (b).

(a)

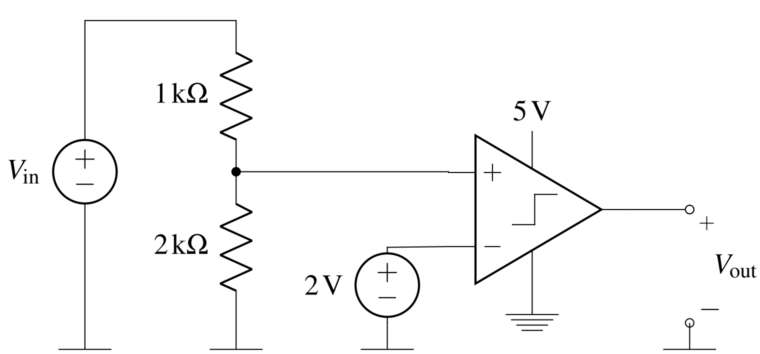


Answer:

When the positive terminal's voltage, V_+, is greater than the negative terminal's voltage, V_-, the value at the positive supply rail, V_{DD},will be output. Likewise, if the negative terminal's voltage, V_-, has a higher voltage then the value at the negative supply rail, V_{SS}, will be output. Since V_- is just the output of a voltage divider with the source $V_{in} = V_+$, it will always have lower absolute value and same polarity as the positive terminal. Thus, the comparator's output will depend only on the sign of the source V_{in}.



(b)



Answer:

$$V_+ = \frac{2\,\text{k}\Omega}{1\,\text{k}\Omega + 2\,\text{k}\Omega} V_{\text{in}} = \frac{2}{3} V_{\text{in}}$$

$$V_- = 2\,\text{V}$$

The comparator will output positive 5V when the voltage divider's output $V_+ > 2\,\text{V}$ and thus when $V_{\text{in}} > 3\,\text{V}$. Otherwise, it will output 0V (ground).