EECS 16A Designing Information Devices and Systems I
Spring 2020 Discussion 6A

1. True or False? (S16 MT Problem)

You only need to write True or False under each subpart.

(a) There exists an invertible $n \times n$ matrix A for which $A^2 = 0$.

Answer: False

Let's left multiply and right multiply A^2 by A^{-1} so we have $A^{-1}AAA^{-1}$. By associativity of matrix multiplication, we have $(A^{-1}A)(AA^{-1}) = I_n I_n = I_n$ where I is the identity matrix. However, if A^2 were 0, then $(A^{-1}A)(AA^{-1}) = A^{-1}A^2A^{-1} = 0$ where 0 is a matrix of all zeros, hence resulting in a contradiction.

(b) If A is an invertible $n \times n$ matrix, then for all vectors $\vec{b} \in \mathbb{R}^n$, the system $A\vec{x} = \vec{b}$ has a unique solution.

Answer: True

If A is invertible, then there is a unique matrix A^{-1}. Left multiply the equation by A^{-1}, and we will have $A^{-1}A\vec{x} = A^{-1}\vec{b} \implies \vec{x} = A^{-1}\vec{b}$, where $\vec{x}$ is a unique vector.

(c) If A and B are invertible $n \times n$ matrices, then the product AB is invertible.

Answer: True

$(AB)^{-1} = B^{-1}A^{-1}$.
Note that $ABB^{-1}A^{-1} = I$ and $B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$

(d) The two vectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ form a basis for the subspace $\text{Span}(\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\})$.

Answer: True.

$\text{Span}(\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\})$ spans the x-y plane in $\mathbb{R}^3$. Since $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ are linearly independent, they form a basis for the x-y plane in $\mathbb{R}^3$ as well.

(e) A set of n linearly dependent vectors in $\mathbb{R}^n$ can span $\mathbb{R}^n$.

Answer: False

A set of n linearly dependent vectors span some subspace of dimension $0 < \dim(A) < n$ in $\mathbb{R}^n$.
Note: It is incorrect to say the set of linearly dependent vectors spans $\mathbb{R}^{n-1}$ for two reasons. First, you don't know what the dimension is of the subspace it spans, which could be less than $n-1$. Second, there is no such thing as $\mathbb{R}^{n-1} \in \mathbb{R}^n$. The vectors are "in" $\mathbb{R}^n$ based on how many elements are in the vector, and a set of vectors spans some subspace (potentially the entire space.)

(f) For all matrices A and B, where A is 5×5 and B is 4×4, it is always the case that $\text{Rank}(A) > \text{Rank}(B)$.

Answer: False

Size does not determine rank! For example, if A was a matrix of all ones rank(A) would be 1. If, on the other hand B was an identity matrix it would have full rank: rank(B) = 4.
You can only claim larger size implies larger rank if you assume the matrices are full rank (pivots in every column, all column vectors are linearly independent from the rest.)

2. Pagerank Review

(a) Consider two linked websites A and B with the following relationship that describes how proportions of visitors move from one to the other:

$$\begin{bmatrix} x_A[k+1] \\ x_B[k+1] \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} x_A[k] \\ x_B[k] \end{bmatrix}$$

Determine if there is a steady state of visitors on each website and if we converge to it. **Answer:** First, find the eigenvalues of the matrix, $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & \frac{5}{6} \end{bmatrix}$, which satisfies $\det(A - \lambda I) = 0$.

$$\det\left(\begin{bmatrix} \frac{1}{2} - \lambda & \frac{1}{6} \\ \frac{1}{2} & \frac{5}{6} - \lambda \end{bmatrix}\right) = (\frac{1}{2} - \lambda)(\frac{5}{6} - \lambda) - \frac{1}{12}$$

$$\lambda^2 - \frac{4}{3}\lambda + \frac{1}{3} = (\lambda - \frac{1}{3})(\lambda - 1) = 0$$

We have eigenvalues of $\lambda_1 = \frac{1}{3}$, and $\lambda_2 = 1$. Since we have an eigenvalue of 1, and the other eigenvector will not grow, we have a steady state that we will also converge to.

(b) Give the fractions of traffic there will be on each site in the long run.
Answer: Now that we know the eigenvalues are $\lambda_1 = \frac{1}{3}$, and $\lambda_2 = 1$, we will want to find the eigenvector corresponding to $\lambda_2 = 1$ which will be the steady state.
We wish to find $\vec{v}_2$ for which $\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & \frac{5}{6} \end{bmatrix} - I\right)\vec{v}_2 = 0$.
Our vector $\vec{v}_2$ is in the nullspace of $\begin{bmatrix} -\frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix}$.
We can inspect that our vector $\vec{v}_2$ is a scalar multiple of $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$.
Once we have $\vec{v}_2$, we can calculate the proportion of the population on each page:
The proportion of the population on A is given by $p_A = \frac{x_{A,ss}}{x_{A,ss}+x_{B,ss}} = \frac{2}{6+2} = \frac{1}{4}$.
The proportion of the population on B is given by $p_B = \frac{x_{B,ss}}{x_{A,ss}+x_{B,ss}} = \frac{6}{6+2} = \frac{3}{4}$.

3. Eigenvectors (F17 MT Problem) Consider a matrix $\mathbf{A} \in \mathbb{R}^{3\times 3}$ with eigenvalues $\lambda = 1, 2, 3$, and corresponding eigenvectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ respectively. Let the matrix $\mathbf{B} = \mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} - 6\mathbf{I}$.

(a) Find $\mathbf{B}\vec{v}$, where $\vec{v}$ is one of the eigenvectors of $\mathbf{A}$.
Hint:

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda - 2)(\lambda - 3)$$

Answer:

$$
\begin{aligned}
\mathbf{B}\vec{v} &= \mathbf{A}^3\vec{v} - 6\mathbf{A}^2\vec{v} + 11\mathbf{A}\vec{v} - 6\mathbf{I}\vec{v} \\
&= \lambda^3\vec{v} - 6\lambda^2\vec{v} + 11\lambda\vec{v} - 6\vec{v} \\
&= (\lambda^3 - 6\lambda^2 + 11\lambda - 6)\vec{v} \\
&= (\lambda-1)(\lambda-2)(\lambda-3)\vec{v}
\end{aligned}
$$

Note that the eigenvalues of $\mathbf{A}$ are $\lambda = 1,2,3$, which means that $\mathbf{B}\vec{v} = \vec{0}$.

(b) Find all the eigenvalues of the matrix $\mathbf{B}$.

Answer:

For any eigenvalue λ of $\mathbf{A}$, where $\mathbf{A}\vec{v} = \lambda\vec{v}$ for some corresponding eigenvector $\vec{v}$,

$$
\begin{aligned}
\mathbf{B}\vec{v} &= \mathbf{A}^3\vec{v} - 6\mathbf{A}^2\vec{v} + 11\mathbf{A}\vec{v} - 6\mathbf{I}\vec{v} \\
&= \lambda^3\vec{v} - 6\lambda^2\vec{v} + 11\lambda\vec{v} - 6\vec{v} \\
&= (\lambda^3 - 6\lambda^2 + 11\lambda - 6)\vec{v} \\
&= (\lambda-1)(\lambda-2)(\lambda-3)\vec{v}
\end{aligned}
$$

This implies that for every eigenvalue λ of $\mathbf{A}$, $(\lambda-1)(\lambda-2)(\lambda-3)$ is an eigenvalue of $\mathbf{B}$ with the same eigenvector.

To find the eigenvalues of $\mathbf{B}$, we plug in every eigenvalue $\lambda = 1,2,3$ of $\mathbf{A}$ to get $\lambda = 0$ in all three cases. Since $\mathbf{B}$ has the same eigenvectors as $\mathbf{A}$, the dimension of the eigenspace of $\mathbf{B}$ corresponding to $\lambda = 0$ is 3, so $\mathbf{B}$ cannot have any other eigenvalues. Therefore, the only eigenvalue of $\mathbf{B}$ is $\lambda = 0$.

(c) Write out the numerical values in the 3×3 matrix $\mathbf{B}$ and justify your answer.

Answer:

Since $\mathbf{A}$ has 3 distinct eigenvalues in $\mathbb{R}^3$, it is diagonalizable, and its eigenvectors form an eigenbasis for $\mathbb{R}^3$. Thus, any $\vec{w} \in \mathbb{R}^3$ can be written as $\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3$. Therefore, for any $\vec{w}$,

$$
\begin{aligned}
\mathbf{B}\vec{w} &= \mathbf{B}\alpha_1\vec{v}_1 + \mathbf{B}\alpha_2\vec{v}_2 + \mathbf{B}\alpha_3\vec{v}_3 \\
&= \alpha_1\mathbf{B}\vec{v}_1 + \alpha_2\mathbf{B}\vec{v}_2 + \alpha_3\mathbf{B}\vec{v}_3 \\
&= \vec{0} + \vec{0} + \vec{0} \\
&= \vec{0}
\end{aligned}
$$

Since $\mathbf{B}\vec{w} = \vec{0}$ for all $\vec{w}$, $\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ must be the zero matrix.

