1. True or False? (S16 MT Problem)

You only need to write True or False under each subpart.

(a) There exists an invertible \( n \times n \) matrix \( A \) for which \( A^2 = 0 \).

\[ \text{Answer: False} \]

Let’s left multiply and right multiply \( A^2 \) by \( A^{-1} \) so we have \( A^{-1} A A^{-1} \). By associativity of matrix multiplication, we have \( (A^{-1}A)(AA^{-1}) = I_n I_n = I_n \) where \( I \) is the identity matrix. However, if \( A^2 \) were 0, then \( (A^{-1}A)(AA^{-1}) = A^{-1}A^2 A^{-1} = 0 \) where 0 is a matrix of all zeros, hence resulting in a contradiction.

(b) If \( A \) is an invertible \( n \times n \) matrix, then for all vectors \( \vec{b} \in \mathbb{R}^n \), the system \( A \vec{x} = \vec{b} \) has a unique solution.

\[ \text{Answer: True} \]

If \( A \) is invertible, then there is a unique matrix \( A^{-1} \). Left multiply the equation by \( A^{-1} \), and we will have \( A^{-1} A \vec{x} = A^{-1} \vec{b} \implies \vec{x} = A^{-1} \vec{b} \), where \( \vec{x} \) is a unique vector.

(c) If \( A \) and \( B \) are invertible \( n \times n \) matrices, then the product \( AB \) is invertible.

\[ \text{Answer: True} \]

\[ (AB)^{-1} = B^{-1}A^{-1} \]

Note that \( ABB^{-1}A^{-1} = I \) and \( B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I \)

(d) The two vectors \( \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \) and \( \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \) form a basis for the subspace \( \text{Span}\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \} \).

\[ \text{Answer: True} \]

\[ \text{Span}\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \} \text{ spans the x-y plane in } \mathbb{R}^3. \] Since \( \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \) and \( \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \) are linearly independent, they form a basis for the x-y plane in \( \mathbb{R}^3 \) as well.

(e) A set of \( n \) linearly dependent vectors in \( \mathbb{R}^n \) can span \( \mathbb{R}^n \).

\[ \text{Answer: False} \]

A set of \( n \) linearly dependent vectors span some subspace of dimension \( 0 < \dim(A) < n \) in \( \mathbb{R}^n \).

\[ \text{Note: It is incorrect to say the set of linearly dependent vectors spans } \mathbb{R}^{n-1} \text{ for two reasons. First, you don’t know what the dimension is of the subspace it spans, which could be less than } n-1. \text{ Second, there is no such thing as } \mathbb{R}^{n-1} \in \mathbb{R}^n. \text{ The vectors are “in” } \mathbb{R}^n \text{ based on how many elements are in the vector, and a set of vectors spans some subspace (potentially the entire space.)} \]

(f) For all matrices \( A \) and \( B \), where \( A \) is \( 5 \times 5 \) and \( B \) is \( 4 \times 4 \), it is always the case that \( \text{Rank}(A) > \text{Rank}(B) \).
2. Pagerank Review

(a) Consider two linked websites $A$ and $B$ with the following relationship that describes how proportions of visitors move from one to the other:

$$
\begin{bmatrix}
    x_A[k+1] \\
    x_B[k+1]
\end{bmatrix} = \begin{bmatrix}
    \frac{1}{3} & \frac{1}{2} \\
    \frac{1}{2} & \frac{5}{6}
\end{bmatrix} \begin{bmatrix}
    x_A[k] \\
    x_B[k]
\end{bmatrix}
$$

Determine if there is a steady state of visitors on each website and if we converge to it. Answer:

First, find the eigenvalues of the matrix, $A = \begin{bmatrix}
    \frac{1}{3} & \frac{1}{2} \\
    \frac{1}{2} & \frac{5}{6}
\end{bmatrix}$, which satisfies $\det(A - \lambda I) = 0$.

$$
\det\left(\begin{bmatrix}
    \frac{1}{2} & \lambda \\
    \frac{5}{6} & -\lambda
\end{bmatrix}\right) = \left(\frac{1}{2} - \lambda\right)\left(\frac{5}{6} - \lambda\right) - \frac{1}{12} = \lambda^2 - \frac{4}{3}\lambda + \frac{1}{3} = (\lambda - \frac{1}{3})(\lambda - 1) = 0
$$

We have eigenvalues of $\lambda_1 = \frac{1}{3}$, and $\lambda_2 = 1$. Since we have an eigenvalue of 1, and the other eigenvector will not grow, we have a steady state that we will also converge to.

(b) Give the fractions of traffic there will be on each site in the long run.

Answer: Now that we know the eigenvalues are $\lambda_1 = \frac{1}{3}$, and $\lambda_2 = 1$, we will want to find the eigenvector corresponding to $\lambda_2 = 1$ which will be the steady state.

We wish to find $\vec{v}_2$ for which $\left(\begin{bmatrix}
    \frac{1}{3} & \frac{1}{2} \\
    \frac{1}{2} & \frac{5}{6}
\end{bmatrix} - I\right)\vec{v}_2 = 0$.

Our vector $\vec{v}_2$ is in the nullspace of $\begin{bmatrix}
    -\frac{1}{3} & \frac{1}{2} \\
    \frac{1}{2} & -\frac{5}{6}
\end{bmatrix}$.

We can inspect that our vector $\vec{v}_2$ is a scalar multiple of $\begin{bmatrix}
    2 \\
    6
\end{bmatrix}$.

Once we have $\vec{v}_2$, we can calculate the proportion of the population on each page:

The proportion of the population on $A$ is given by $p_A = \frac{x_{A,ss}}{x_{A,ss} + x_{B,ss}} = \frac{2}{9+2} = \frac{1}{4}$.

The proportion of the population on $B$ is given by $p_B = \frac{x_{B,ss}}{x_{A,ss} + x_{B,ss}} = \frac{6}{9+2} = \frac{3}{4}$.

3. Eigenvectors (F17 MT Problem) Consider a matrix $A \in \mathbb{R}^{3 \times 3}$ with eigenvalues $\lambda = 1, 2, 3$, and corresponding eigenvectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ respectively. Let the matrix $B = A^3 - 6A^2 + 11A - 6I$.

(a) Find $B\vec{v}$, where $\vec{v}$ is one of the eigenvectors of $A$.

Hint: $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda - 2)(\lambda - 3)$

Answer:
\[ \mathbf{Bv} = \mathbf{A}^3 \mathbf{v} - 6 \mathbf{A}^2 \mathbf{v} + 11 \mathbf{A} \mathbf{v} - 6 \mathbf{Iv} \]
\[ = \lambda^3 \mathbf{v} - 6 \lambda^2 \mathbf{v} + 11 \lambda \mathbf{v} - 6 \mathbf{v} \]
\[ = (\lambda^3 - 6 \lambda^2 + 11 \lambda - 6) \mathbf{v} \]
\[ = (\lambda - 1)(\lambda - 2)(\lambda - 3) \mathbf{v} \]

Note that the eigenvalues of \( \mathbf{A} \) are \( \lambda = 1, 2, 3 \), which means that \( \mathbf{Bv} = \mathbf{0} \).

(b) Find all the eigenvalues of the matrix \( \mathbf{B} \).

**Answer:**

For any eigenvalue \( \lambda \) of \( \mathbf{A} \), where \( \mathbf{Av} = \lambda \mathbf{v} \) for some corresponding eigenvector \( \mathbf{v} \),

\[ \mathbf{Bv} = \mathbf{A}^3 \mathbf{v} - 6 \mathbf{A}^2 \mathbf{v} + 11 \mathbf{A} \mathbf{v} - 6 \mathbf{Iv} \]
\[ = \lambda^3 \mathbf{v} - 6 \lambda^2 \mathbf{v} + 11 \lambda \mathbf{v} - 6 \mathbf{v} \]
\[ = (\lambda^3 - 6 \lambda^2 + 11 \lambda - 6) \mathbf{v} \]
\[ = (\lambda - 1)(\lambda - 2)(\lambda - 3) \mathbf{v} \]

This implies that for every eigenvalue \( \lambda \) of \( \mathbf{A} \), \((\lambda - 1)(\lambda - 2)(\lambda - 3) \) is an eigenvalue of \( \mathbf{B} \) with the same eigenvector.

To find the eigenvalues of \( \mathbf{B} \), we plug in every eigenvalue \( \lambda = 1, 2, 3 \) of \( \mathbf{A} \) to get \( \lambda = 0 \) in all three cases. Since \( \mathbf{B} \) has the same eigenvectors as \( \mathbf{A} \), the dimension of the eigenspace of \( \mathbf{B} \) corresponding to \( \lambda = 0 \) is 3, so \( \mathbf{B} \) cannot have any other eigenvalues. Therefore, the only eigenvalue of \( \mathbf{B} \) is \( \lambda = 0 \).

(c) Write out the numerical values in the \( 3 \times 3 \) matrix \( \mathbf{B} \) and justify your answer.

**Answer:**

Since \( \mathbf{A} \) has 3 distinct eigenvalues in \( \mathbb{R}^3 \), it is diagonalizable, and its eigenvectors form an eigenbasis for \( \mathbb{R}^3 \). Thus, any \( \mathbf{w} \in \mathbb{R}^3 \) can be written as \( \mathbf{w} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 \). Therefore, for any \( \mathbf{w} \),

\[ \mathbf{Bw} = \mathbf{B} \alpha_1 \mathbf{v}_1 + \mathbf{B} \alpha_2 \mathbf{v}_2 + \mathbf{B} \alpha_3 \mathbf{v}_3 \]
\[ = \alpha_1 \mathbf{Bv}_1 + \alpha_2 \mathbf{Bv}_2 + \alpha_3 \mathbf{Bv}_3 \]
\[ = \mathbf{0} + \mathbf{0} + \mathbf{0} \]
\[ = \mathbf{0} \]

Since \( \mathbf{Bw} = \mathbf{0} \) for all \( \mathbf{w} \), \( \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \) must be the zero matrix.