EECS 16A Designing Information Devices and Systems I
Spring 2020 Discussion 5B

1. Steady and Unsteady States

(a) You're given the matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

Which generates the next state of a physical system from its previous state: $\vec{x}[k+1] = \mathbf{M}\vec{x}[k]$. ($\vec{x}$ could describe either people or water.) Find the eigenspaces associated with the following eigenvalues:

i. span($\vec{v}_1$), associated with $\lambda_1 = 1$
ii. span($\vec{v}_2$), associated with $\lambda_2 = 2$
iii. span($\vec{v}_3$), associated with $\lambda_3 = \frac{1}{2}$

Answer:

i. $\lambda = 1$:

$$\left[\begin{array}{c|c} \mathbf{M}-\mathbf{I} & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_1 = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}$$

ii. $\lambda = 2$

$$\left[\begin{array}{c|c} \mathbf{M}-2\mathbf{I} & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} \frac{-3}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -3 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_2 = \beta \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \beta \in \mathbb{R}$$

iii. $\lambda = \frac{1}{2}$

$$\left[\begin{array}{c|c} \mathbf{M}-\frac{1}{2}\mathbf{I} & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -2 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\vec{v}_3 = \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \gamma \in \mathbb{R}$$

(b) Define $\vec{x} = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$, a linear combination of the eigenvectors. For each of the cases in the table, determine if

$$\lim_{n\to\infty} \mathbf{M}^n \vec{x}$$

converges. If it does, what does it converge to?

α	β	γ	Converges?	$\lim_{n\to\infty} \mathbf{M}^n \vec{x}$
0	0	$\neq 0$		
0	$\neq 0$	0		
0	$\neq 0$	$\neq 0$		
$\neq 0$	0	0		
$\neq 0$	0	$\neq 0$		
$\neq 0$	$\neq 0$	0		
$\neq 0$	$\neq 0$	$\neq 0$		

Answer:

$$\begin{aligned}\mathbf{M}^n \vec{x} &= \mathbf{M}^n(\alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3) \\ &= \alpha\mathbf{M}^n\vec{v}_1 + \beta\mathbf{M}^n\vec{v}_2 + \gamma\mathbf{M}^n\vec{v}_3 \\ &= 1^n\alpha\vec{v}_1 + 2^n\beta\vec{v}_2 + \left(\frac{1}{2}\right)^n \gamma\vec{v}_3\end{aligned}$$

α	β	γ	Converges?	$\lim_{n\to\infty} \mathbf{M}^n \vec{x}$
0	0	$\neq 0$	Yes	$\vec{0}$
0	$\neq 0$	0	No	-
0	$\neq 0$	$\neq 0$	No	-
$\neq 0$	0	0	Yes	$\alpha\vec{v}_1$
$\neq 0$	0	$\neq 0$	Yes	$\alpha\vec{v}_1$
$\neq 0$	$\neq 0$	0	No	-
$\neq 0$	$\neq 0$	$\neq 0$	No	-

2. Rabbits, Foxes, and the Circle of Life

If rabbits are such notoriously fast breeders, why haven't we all been crushed under a (warm, comfortable) mountain of rabbits by now? Well, consider the hungry foxes...

Let's examine the case of Tilden Park, circa 1000 CE. This vast beautiful space is initially filled with 200 foxes and 1000 rabbits. Since rabbits like to feast on the pleantiful greenery, the population of rabbits grows by 10% each month. Every month, 40% of the foxes either die or leave the park. The population of foxes increases by 20% of the population of rabbits each month. Similarly the population of rabbits decreases by 20% of the fox population each month. This can be summarized by the system shown below, where $f[t]$ and $r[t]$ represent the number of foxes and rabbits in the park each month t.

$$\begin{bmatrix} f[t+1] \\ r[t+1] \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 \\ -0.2 & 1.1 \end{bmatrix} \begin{bmatrix} f[t] \\ r[t] \end{bmatrix}$$

In this problem, we will use linear algebra to explore the predator-prey relationship to figure out if we should be submerged in rabbits, on the run from armies of foxes, or in some peaceful equilibrium state.

(a) We want to know what will happen to the populations of the two species as time goes on. Will the population numbers converge?

Note: You do not need to find what the populations converge to, just whether they converge or not. You may or may not find it useful to know that $1.7^2 = 2.89$.

Answer:

Finding the eigenvalues of the system allows us to get a sense of the state transformation after many time steps.

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \det\left(\begin{bmatrix} 0.6-\lambda & 0.2 \\ -0.2 & 1.1-\lambda \end{bmatrix}\right) = 0$$

$$\begin{aligned} \lambda^2 - 1.7\lambda + 0.7 &= 0 \\ (\lambda - 1)(\lambda - 0.7) &= 0 \\ \lambda &= 0.7, 1 \end{aligned}$$

We find the eigenvalues of $\lambda = 1$ and $\lambda = 0.7$. Since we have an eigenvalue $\lambda = 1$ together with an eigenvalue $\lambda = 0.7 < 1$, we know that the system will converge to some stable values.

Common Mistakes:

It is not sufficent to show that this matrix has the eigenvalue 1; you must show that all other eigenvalues are less than 1.

(b) Assuming that there is some known total populaton of foxes and rabbits in the year 2000, **calculate what fraction of that total population is rabbits.** You can assume that 1000 years is a good approximation for an infinite amount of time.

Answer:

To find what the state vector converges to, we need to find the eigenvectors. It helps to first convert the original matrix into fraction form before proceeding.

By plugging the eigenvalues from the previous part into the equation $\mathbf{A} - \lambda \mathbf{I} = \vec{0}$ and solving using Gaussian elimination, we find the eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ corresponding to $\lambda = 1$ and the eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ corresponding to $\lambda = 0.7$.

At steady state, the component of the intial state corresponding to the eigenvector with eigenvalue less than 1 will converge to zero. Thus, the steady state vector is just a scaled version of the eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. From this, we know that $\frac{2}{3}$ of the population are rabbits.

(c) In the far future, a curious child digs up a strange fossil in the park. It seems like Ancient Dino-foxes once inhabited the park! Using advanced future Zoologic-Mathematics, graduate students from the University of MegaCalifornia, Berkeley derive the following eigenvalue/eigenvector pairs describing the species interactions from the fossils:

$$\left(\lambda_1 = 1, \vec{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right), \left(\lambda_2 = 0.5, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

Reconstruct the state transition matrix.

Answer:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1}$$

We have four unknowns, a, b, c, d and four equations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \lambda_1 \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \lambda_2 \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} \tag{3}$$

Forming the system of equations,

$$\begin{bmatrix} v_{11} & v_{12} & 0 & 0 \\ 0 & 0 & v_{11} & v_{12} \\ v_{21} & v_{22} & 0 & 0 \\ 0 & 0 & v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \lambda_1 v_{11} \\ \lambda_1 v_{12} \\ \lambda_2 v_{21} \\ \lambda_2 v_{22} \end{bmatrix} \tag{4}$$

Plugging in and solving you get:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{2}{3} \end{bmatrix} \tag{5}$$

The associated system of equations is given by:

$$\begin{cases} f[t+1] = \dfrac{5}{6} f[t] - \dfrac{1}{3} r[t] \\ r[t+1] = -\dfrac{1}{6} f[t] + \dfrac{2}{3} r[t] \end{cases}$$

It seems that Dino-Rabbits actually ate Dino-Foxes!

3. Proofs

(a) Given that $\det(\mathbf{A}) = \det(\mathbf{A}^T)$, show that $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalues.

Answer: The eigenvalues of $\mathbf{A}$ satisfy $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$. However, if determinants of a matrix and its transpose are equal, then it is the case that $\det(\mathbf{A} - \lambda \mathbf{I}) = \det\left((\mathbf{A} - \lambda \mathbf{I})^T\right) = \det(\mathbf{A}^T - \lambda \mathbf{I})$. For the choice of λ that makes $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$, we have that $\det(\mathbf{A}^T - \lambda \mathbf{I}) = 0$ as well. So the eigenvalues of $\mathbf{A}$ are the same as those of $\mathbf{A}^T$.

