EECS 16A Designing Information Devices and Systems I
Spring 2020 Discussion 5A

Definition: If $A\vec{x} = \lambda\vec{x}$ for $\vec{x} \neq \vec{0}$, then λ is called an eigenvalue of A, and $\vec{x}$ is called an eigenvector. The set of all vectors $\vec{x}$ satisfying $A\vec{x} = \lambda\vec{x}$ for some λ is called the eigenspace of λ.

1. Mechanical Eigenvalues and Eigenvectors

In each part, find the eigenvalues of the matrix $\mathbf{M}$ and the associated eigenvectors. State if the inverse of $\mathbf{M}$ exists.

(a) $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}\right) = 0$$

$$\begin{aligned} -\lambda(-3-\lambda) + 2 &= 0 \\ \lambda^2 + 3\lambda + 2 &= 0 \\ \lambda &= -1, -2 \end{aligned}$$

$\lambda = -1$:

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$x + y = 0 \implies y = -x \implies \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The eigenspace for $\lambda = -1$ is span$\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$.

$\lambda = -2$:

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$2x + y = 0 \implies y = -2x \implies \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The eigenspace for $\lambda = -2$ is span$\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\}$.

Note that we have no non-zero eigenvalues, the columns of A are linearly independent, and the determinant of A is non-zero (evaluate our polynomial in λ at $\lambda = 0$). Any of these are equivalent conditions for saying that a square matrix is invertible.

(b) $\mathbf{M} = \begin{bmatrix} -2 & 4 \\ -4 & 8 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -2-\lambda & 4 \\ -4 & 8-\lambda \end{bmatrix}\right) = 0$$

$$(-2-\lambda)(8-\lambda)+16 = 0$$
$$\lambda^2 - 6\lambda = 0 \implies \lambda(\lambda-6) = 0$$
$$\lambda = 0, 6$$

$\lambda = 0$:

$$\begin{bmatrix} -2 & 4 \\ -4 & 8 \end{bmatrix} \sim \begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix}$$

$$-2x + 4y = 0 \implies y = \frac{1}{2}x \implies \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The eigenspace for $\lambda = 0$ is span$\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$.

$\lambda = 6$:

$$\begin{bmatrix} -2-6 & 4 \\ 4 & 8-6 \end{bmatrix} = \begin{bmatrix} -8 & 4 \\ -4 & 2 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$-2x + y = 0 \implies y = 2x \implies \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The eigenspace for $\lambda = 6$ is span$\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$.

Matrix $\mathbf{M}$ has linearly dependent columns, therefore the inverse $\mathbf{M}^{-1}$ does not exist.

(c) $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 0 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

$$\lambda^2 = 0$$
$$\lambda = 0(\times 2)$$

$\lambda = 0$:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$y = 0, x \text{ is free} \implies \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The eigenspace for $\lambda = 0$ is span$\{\begin{bmatrix}1\\0\end{bmatrix}\}$. Note even though $\lambda = 0$ is a eigenvalue with multiplicity 2 (occurs as a root twice for the characteristic polynomial), the dimension of its eigenspace is only 1. In general, it is not possible to find as many linearly independent vectors for as many times a specific eigenvalue occurs for a matrix.

Matrix $\mathbf{M}$ has a zero column (linearly dependent columns), therefore the inverse $\mathbf{M}^{-1}$ does not exist.

(d) $\mathbf{M} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$.

Answer: Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} - \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}\right) = \det\left(\begin{bmatrix}-\lambda & -1\\1 & -\lambda\end{bmatrix}\right) = 0$$
$$\lambda^2 + 1 = 0$$

From the above equation, we know that the eigenvalues are $\lambda = i$ and $\lambda = -i$.

For the eigenvalue $\lambda = i$:

$$(\mathbf{M} - i\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} - i\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix}-i & -1\\1 & -i\end{bmatrix}\vec{x} = \vec{0}$$

which is simply $x_1 = ix_2$ and x_2 is free or equivalently $\begin{bmatrix}ix_2\\x_2\end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix}i\\1\end{bmatrix}\right\}$.

For the eigenvalue $\lambda = -i$:

$$(\mathbf{M} + i\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} + i\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} + \begin{bmatrix}i & 0\\0 & i\end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix}i & -1\\1 & i\end{bmatrix}\vec{x} = \vec{0}$$

which is simply $x_1 = -ix_2$ and x_2 is free or equivalently $\begin{bmatrix}-ix_2\\x_2\end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix}-i\\1\end{bmatrix}\right\}$.

(e) **(PRACTICE)** $\mathbf{M} = \begin{bmatrix}1 & 0\\0 & 9\end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix}1 & 0\\0 & 9\end{bmatrix} - \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}\right) = \det\left(\begin{bmatrix}1-\lambda & 0\\0 & 9-\lambda\end{bmatrix}\right) = 0$$

The determinant of a diagonal matrix is the product of the entries.

$$(1-\lambda)(9-\lambda)=0$$

From the above equation, we know that the eigenvalues are $\lambda = 1$ and $\lambda = 9$.
For the eigenvalue $\lambda = 1$:

$$\begin{aligned}(\mathbf{M}-1\mathbf{I})\vec{x}&=\vec{0}\\ \left(\begin{bmatrix}1&0\\0&9\end{bmatrix}-1\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\vec{x}&=\vec{0}\\ \begin{bmatrix}0&0\\0&8\end{bmatrix}\vec{x}&=\vec{0}\end{aligned}$$

which is simply $x_2 = 0$ or equivalently $\begin{bmatrix}x_1\\0\end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$.
For the eigenvalue $\lambda = 9$:

$$\begin{aligned}(\mathbf{M}-9\mathbf{I})\vec{x}&=\vec{0}\\ \left(\begin{bmatrix}1&0\\0&9\end{bmatrix}-9\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\vec{x}&=\vec{0}\\ \left(\begin{bmatrix}1&0\\0&9\end{bmatrix}-\begin{bmatrix}9&0\\0&9\end{bmatrix}\right)\vec{x}&=\vec{0}\\ \begin{bmatrix}-8&0\\0&0\end{bmatrix}\vec{x}&=\vec{0}\end{aligned}$$

which is simply $x_1 = 0$ or equivalently $\begin{bmatrix}0\\x_2\end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$.
The matrix is invertible.

2. Eigenvalues and Special Matrices – Visualization

An eigenvector $\vec{v}$ belonging to a square matrix $\mathbf{A}$ is a nonzero vector that satisfies

$$\mathbf{A}\vec{v}=\lambda\vec{v}$$

where λ is a a scalar known as the **eigenvalue** corresponding to eigenvector $\vec{v}$.

The following parts don't require knowledge about how to find eigenvalues. Answer each part by reasoning about the matrix at hand.

(a) Does the identity matrix in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

Answer:

Multiplying the identity matrix with any vector in $\mathbb{R}^n$ produces the same vector, that is, $\mathbf{I}\vec{x} = \vec{x} = 1\cdot\vec{x}$. Therefore, $\lambda = 1$. Since $\vec{x}$ can be any vector in $\mathbb{R}^n$, the corresponding eigenvectors are all vectors in $\mathbb{R}^n$.

(b) Does a diagonal matrix $\begin{bmatrix}d_1&0&0&\cdots&0\\0&d_2&0&\cdots&0\\0&0&d_3&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&d_n\end{bmatrix}$ in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

Answer:

Since the matrix is diagonal, multiplying the diagonal matrix with any standard basis vector $\vec{e}_i$ produces $d_i\vec{e}_i$, that is, $\mathbf{D}\vec{e}_i = d_i\vec{e}_i$. Therefore, the eigenvalues are the diagonal entries d_i of $\mathbf{D}$, and the corresponding eigenvector associated with $\lambda = d_i$ is the standard basis vector $\vec{e}_i$.

(c) Does a rotation matrix in $\mathbb{R}^2$ have any eigenvalues $\lambda \in \mathbb{R}$?

Answer:

There are three cases:

i. Rotation by $0°$ (more accurately, any integer multiple of $360°$), which yields a rotation matrix $\mathbf{R} = \mathbf{I}$: This will have one eigenvalue of $+1$ because it doesn't affect any vector ($\mathbf{R}\vec{x} = \vec{x}$). The eigenspace associated with it is $\mathbb{R}^2$.

ii. Rotation by $180°$ (more accurately, any angle of $180° + n \cdot 360°$ for integer n), which yields a rotation matrix $\mathbf{R} = -\mathbf{I}$: This will have one eigenvalue of -1 because it "flips" any vector ($\mathbf{R}\vec{x} = -\vec{x}$). The eigenspace associated with it is $\mathbb{R}^2$.

iii. Any other rotation: there aren't any real eigenvalues. The reason is, if there were any real eigenvalue $\lambda \in \mathbb{R}$ for a non-trivial rotation matrix, it means that we can get $\mathbf{R}\vec{x} = \lambda\vec{x}$ for some $\vec{x} \neq \vec{0}$, which means that by rotating a vector, we scaled it. This is a contradiction (again, unless $\mathbf{R} = \mathbf{I}$). Refer to Figure 1 for a visualization.

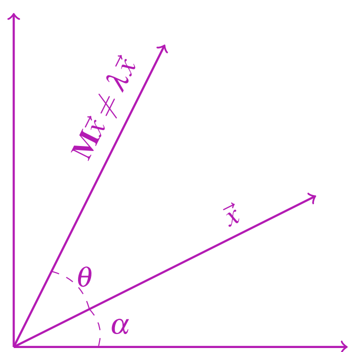

Figure 1: Rotation will never scale any non-zero vector (by a real number) unless it is rotation by an integer multiple of $360°$ (identity matrix) or the rotation angle is $\theta = 180° + n \cdot 360°$ for any integer n ($-\mathbf{I}$).

(d) Does a reflection matrix in $\mathbb{R}^{2\times 2}$, where the reflection is around any line passing through the origin, have any eigenvalues $\lambda \in \mathbb{R}$?

Answer:

Yes, both $+1$ and -1. Why? Reflecting any vector that is on the reflection axis will not affect it (eigenvalue $+1$). Reflecting any vector orthogonal (perpendicular) to the reflection axis will just "flip it/negate it" (eigenvalue -1). In other words, the set of vectors that lie along the axis of reflection is the eigenspace associated with the eigenvalue $+1$ and the set of vectors orthogonal to the axis of reflection is the eigenspace associated with the eigenvalue -1. Refer to Figure 2 for a visualization.

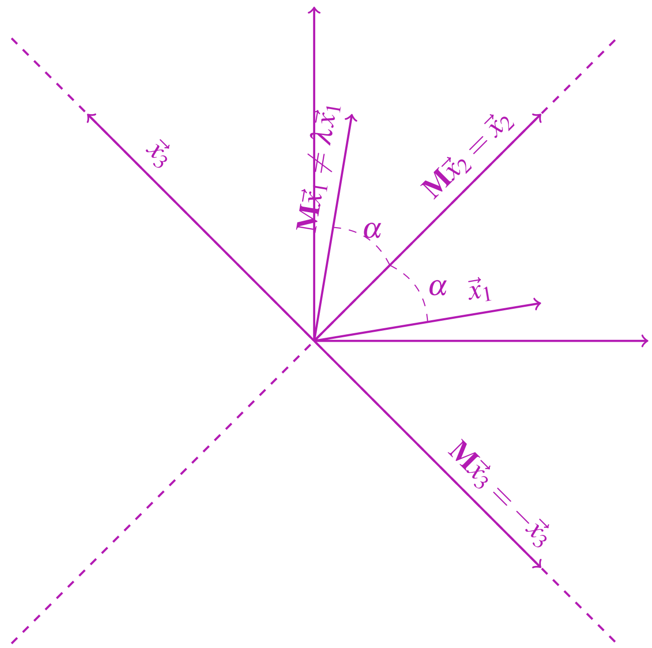

Figure 2: Reflection will scale vectors on the reflection axis (by $+1$) and orthogonal to it (by -1).

(e) If a matrix $\mathbf{M}$ has an eigenvalue $\lambda = 0$, what does this say about its null space? What does this say about the solutions of the system of linear equations $\mathbf{M}\vec{x} = \vec{b}$?

Answer:

$N(A)$ is not just $\vec{0}$ as we have some $\vec{x} \neq \vec{0}$ satisfying $A\vec{x} = \lambda\vec{x}$. Another way we can state this is that $\dim(N(A)) > 0$. $\mathbf{M}\vec{x} = \vec{b}$ is either inconsistent or has infinitely many solutions.

(f) **(Practice)** Does the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

Answer:

Note that the matrix has linearly dependent columns. Therefore, according to part (e), one eigenvalue is $\lambda = 0$. The corresponding eigenvector, which is equivalent to the basis vector for the null space, is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The other eigenvalue is, by inspection, $\lambda = 1$ with the corresponding eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ because $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

3. Steady State Reservoir Levels

We have 3 reservoirs: A, B and C. The pumps system between the reservoirs is depicted in Figure 3.

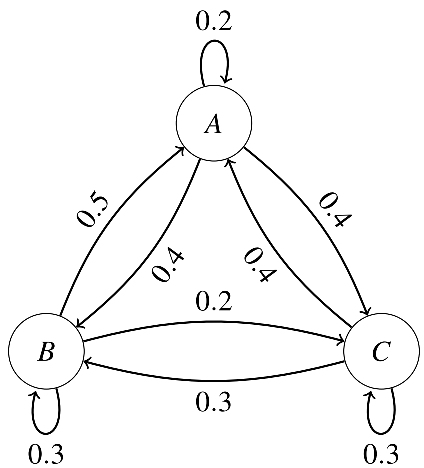

Figure 3: Reservoir pumps system.

(a) Write out the transition matrix $\mathbf{T}$ representing the pumps system.

Answer:

$$\mathbf{T} = \begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix}$$

(b) You are told that $\lambda_1 = 1$, $\lambda_2 = \frac{-\sqrt{2}-1}{10}$, $\lambda_3 = \frac{\sqrt{2}-1}{10}$ are the eigenvalues of $\mathbf{T}$. Find a steady state vector $\vec{x}$, i.e. a vector such that $T\vec{x} = \vec{x}$.

Answer:

We know $\lambda_1 = 1$ is the eigenvalue corresponding to the steady state eigenvector. Therefore,

$$T\vec{x} = 1\vec{x} = \lambda_1\vec{x} \Rightarrow \vec{x} \in N(\mathbf{T} - 1\cdot\mathbf{I}) = N\left(\begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) =$$

$$N\left(\begin{bmatrix} -0.8 & 0.5 & 0.4 \\ 0.4 & -0.7 & 0.3 \\ 0.4 & 0.2 & -0.7 \end{bmatrix}\right)$$

In order to row reduce $\mathbf{T} - 1\cdot\mathbf{I}$ we use Gaussian elimination:

$$\begin{bmatrix} -0.8 & 0.5 & 0.4 \\ 0.4 & -0.7 & 0.3 \\ 0.4 & 0.2 & -0.7 \end{bmatrix} \overbrace{\Rightarrow}^{R_2 \leftarrow R_1 + 2R_2} \begin{bmatrix} -0.8 & 0.5 & 0.4 \\ 0 & -0.9 & 1 \\ 0.4 & 0.2 & -0.7 \end{bmatrix} \overbrace{\Rightarrow}^{R_3 \leftarrow R_1 + 2R_3} \begin{bmatrix} -0.8 & 0.5 & 0.4 \\ 0 & -0.9 & 1 \\ 0 & 0.9 & -1 \end{bmatrix}$$

$$\overbrace{\Rightarrow}^{R_3 \leftarrow R_2 + R_3} \begin{bmatrix} -0.8 & 0.5 & 0.4 \\ 0 & -0.9 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is a vector describing the steady state, then we can set x_3 to be the free variable. Therefore, we can write:

$x_3 = a$, $0.9x_2 = a$ and $x_1 = \frac{0.5x_2+0.4x_3}{0.8} \Rightarrow x_3 = a$, $x_2 = \frac{10}{9}a$ and $x_1 = \frac{43}{36}a$

which means that our steady state vector is of the form $\begin{bmatrix} \frac{43}{36}\alpha \\ \frac{10}{9}\alpha \\ \alpha \end{bmatrix}, \alpha \in \mathbb{R}$.

