EECS 16A Designing Information Devices and Systems I
Spring 2020 Discussion 4B

1. Exploring Column Spaces and Null Spaces

- The **column space** is the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

i. What is the column space of **A**? What is its dimension?

ii. What is the null space of **A**? What is its dimension?

iii. Are the column spaces of the row reduced matrix **A** and the original matrix **A** the same?

iv. Do the columns of **A** form a basis for $\mathbb{R}^2$? Why or why not?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Answer:

Column space: span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Null space: span $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

The matrix is already row reduced. The column spaces of the row reduced matrix and the original matrix are the same.

Not a basis for $\mathbb{R}^2$.

(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

Answer:

Column space: span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Null space: span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

The two column spaces are not the same.

Not a basis for $\mathbb{R}^2$.

(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

Answer:

Column space: $\mathbb{R}^2$

Null space: span $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

The two column spaces are the same as the column span $\mathbb{R}^2$.

This is a basis for $\mathbb{R}^2$.

(d) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$

Answer:

Column space: span $\left\{ \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} \right\}$

Null space: span $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

The two column spaces are not the same.
Not a basis for $\mathbb{R}^2$.

(e) $\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

Answer:

i. The columnspace of the columns is $\mathbb{R}^2$. The columns of $\mathbf{A}$ do not form a basis for $\mathbb{R}^2$. This is because the columns of $\mathbf{A}$ are linearly dependent.

ii. The following algorithm can be used to solve for the null space of a matrix. The procedure is essentially solving the matrix-vector equation $\mathbf{A}\vec{x} = \vec{0}$ by performing Gaussian elimination on $\mathbf{A}$. We start by performing Gaussian elimination on matrix $\mathbf{A}$ to get the matrix into upper-triangular form.

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{7}{2} \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \text{reduced row echelon form}$$

$$x_1 + \frac{1}{2}x_3 - \frac{7}{2}x_4 = 0$$
$$x_2 + \frac{5}{2}x_3 + \frac{1}{2}x_4 = 0$$
$$x_3 \text{ is free and } x_4 \text{ is free}$$

Now let $x_3 = s$ and $x_4 = t$. Then we have:

$$x_1 + \frac{1}{2}s - \frac{7}{2}t = 0$$
$$x_2 + \frac{5}{2}s + \frac{1}{2}t = 0$$

Now writing all the unknowns (x_1, x_2, x_3, x_4) in terms of the dummy variables:

$$x_1 = -\frac{1}{2}s + \frac{7}{2}t$$
$$x_2 = -\frac{5}{2}s - \frac{1}{2}t$$
$$y = s$$
$$z = t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s + \frac{7}{2}t \\ -\frac{5}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s \\ -\frac{5}{2}s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{7}{2}t \\ -\frac{1}{2}t \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

So every vector in the nullspace of **A** can be written as follows:

$$\text{Nullspace}(\mathbf{A}) = s \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Therefore the nullspace of **A** is

$$\text{span}\left\{ \left[\begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right] \right\}$$

A has a 2-dimensional null space.

iii. In this case, the column space of the row reduced matrix is also $\mathbb{R}^2$, but this need not be true in general.

iv. No, the columns of **A** do not form a basis for $\mathbb{R}^2$.

2. Identifying a Basis

Does each of these sets of vectors describe a basis for $\mathbb{R}^3$? If the vectors do not form a basis for $\mathbb{R}^3$, can they be thought of as a basis for some other vector space? If so, write an expression describing this vector space.

$$V_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad V_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \quad V_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Answer:

- V_1: The vectors are linearly independent, but they are not a basis for $\mathbb{R}^3$, because you cannot construct all vectors in $\mathbb{R}^3$ using these vectors. Instead, they are a basis for some 2-dimensional subspace of $\mathbb{R}^3$. This subspace can be described by span$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.
- V_2: Yes, the vectors are linearly independent and will form a basis for $\mathbb{R}^3$. To check that the vectors are linearly independent, you should do Gaussian Elimination of the matrix of the columns: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. Check that you can get all the way to identity, i.e. the system has a unique solution.
- V_3: No, $\vec{v}_2 + \vec{v}_3 = \vec{v}_1$, so the vectors are linearly dependent. Hence, they cannot form a basis for any vector space of any dimension.

3. Subspaces, Bases, and Dimension

For each of the sets $U_i \subseteq \mathbb{R}^3$ defined below, state whether it is a subspace or not. If it is a subspace, find a basis for it and state the dimension.

(a) $U_1 = \left\{ \begin{bmatrix} 2(x+y) \\ x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

Answer: U_1 is a subspace described by the basis $\mathscr{B}_1$, where

$$\mathscr{B}_1 = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The subspace has dimension 2, since there are 2 basis vectors.

(b) $U_2 = \left\{ \begin{bmatrix} x \\ y \\ z+1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$

Answer: U_2 is a subspace (in fact $\mathbb{R}^3$) of dimension 3, and a basis is the natural basis.

(c) $U_3 = \left\{ \begin{bmatrix} x \\ y \\ x+1 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

Answer: U_3 is not a subspace since it does not contain the zero vector and is therefore not closed under scalar multiplication.

(d) $U_4 = \left\{ \begin{bmatrix} x \\ y \\ (x+y)^2 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

Answer: U_4 is not a subspace, since it is not closed under scalar multiplication or vector addition and thus fails to satisfy the definition of a vector space. As an example, we can see that vector $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is in the space, but vector $\vec{v}_2 = 2\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ is not, so the scalar-multiplication property doesn't hold; a similar argument can be made for vector addition.