1. Exploring Column Spaces and Null Spaces

- The **column space** is the *span* of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

i. What is the column space of \( A \)? What is its dimension?

ii. What is the null space of \( A \)? What is its dimension?

iii. Are the column spaces of the row reduced matrix \( A \) and the original matrix \( A \) the same?

iv. Do the columns of \( A \) form a basis for \( \mathbb{R}^2 \)? Why or why not?

(a) \[
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
\end{bmatrix}
\]

**Answer:**

Column space: span \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)

Null space: span \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)

The matrix is already row reduced. The column spaces of the row reduced matrix and the original matrix are the same.

Not a basis for \( \mathbb{R}^2 \).

(b) \[
\begin{bmatrix}
0 & 1 \\
0 & 1 \\
\end{bmatrix}
\]

**Answer:**

Column space: span \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \)

Null space: span \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)

The two column spaces are not the same.

Not a basis for \( \mathbb{R}^2 \).

(c) \[
\begin{bmatrix}
1 & 2 \\
-1 & 1 \\
\end{bmatrix}
\]

**Answer:**

Column space: \( \mathbb{R}^2 \)

Null space: span \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \)

The two column spaces are the same as the column span \( \mathbb{R}^2 \).

This is a basis for \( \mathbb{R}^2 \).
(d) \[
\begin{bmatrix}
-2 & 4 \\
3 & -6
\end{bmatrix}
\]

**Answer:**

Column space: \( \text{span} \left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \right\} \)

Null space: \( \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \)

The two column spaces are not the same. Not a basis for \( \mathbb{R}^2 \).

(e) \[
\begin{bmatrix}
1 & -1 & -2 & -4 \\
1 & 1 & 3 & -3
\end{bmatrix}
\]

**Answer:**

i. The columnspace of the columns is \( \mathbb{R}^2 \). The columns of \( A \) do not form a basis for \( \mathbb{R}^2 \). This is because the columns of \( A \) are linearly dependent.

ii. The following algorithm can be used to solve for the null space of a matrix. The procedure is essentially solving the matrix-vector equation \( Ax = 0 \) by performing Gaussian elimination on \( A \).

We start by performing Gaussian elimination on matrix \( A \) to get the matrix into upper-triangular form.

\[
\begin{bmatrix}
1 & -1 & -2 & -4 \\
1 & 1 & 3 & -3
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -1 & -2 & -4 \\
0 & 2 & 5 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -1 & -2 & -4 \\
0 & 1 & 5/2 & 1/2
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 1/2 & -7/2 \\
0 & 1 & 5/2 & 1/2
\end{bmatrix}
\]

reduced row echelon form

\[x_1 + \frac{1}{2}x_3 - \frac{7}{2}x_4 = 0\]
\[x_2 + \frac{5}{2}x_3 + \frac{1}{2}x_4 = 0\]

\(x_3\) is free and \(x_4\) is free

Now let \( x_3 = s \) and \( x_4 = t \). Then we have:

\[x_1 + \frac{1}{2}s - \frac{7}{2}t = 0\]
\[x_2 + \frac{5}{2}s + \frac{1}{2}t = 0\]

Now writing all the unknowns \((x_1, x_2, x_3, x_4)\) in terms of the dummy variables:

\[x_1 = -\frac{1}{2}s + \frac{7}{2}t\]
\[x_2 = -\frac{5}{2}s - \frac{1}{2}t\]
\[y = s\]
\[z = t\]
So every vector in the nullspace of $A$ can be written as follows:

$$\text{Nullspace}(A) = s \begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Therefore the nullspace of $A$ is

$$\text{span} \left\{ \begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$$

$A$ has a 2-dimensional null space.

iii. In this case, the column space of the row reduced matrix is also $\mathbb{R}^2$, but this need not be true in general.

iv. No, the columns of $A$ do not form a basis for $\mathbb{R}^2$.

2. Identifying a Basis

Does each of these sets of vectors describe a basis for $\mathbb{R}^3$? If the vectors do not form a basis for $\mathbb{R}^3$, can they be thought of as a basis for some other vector space? If so, write an expression describing this vector space.

$$V_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad V_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\} \quad V_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Answer:

- $V_1$: The vectors are linearly independent, but they are not a basis for $\mathbb{R}^3$, because you cannot construct all vectors in $\mathbb{R}^3$ using these vectors. Instead, they are a basis for some 2-dimensional subspace of $\mathbb{R}^3$.

  This subspace can be described by $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

- $V_2$: Yes, the vectors are linearly independent and will form a basis for $\mathbb{R}^3$. To check that the vectors are linearly independent, you should do Gaussian Elimination of the matrix of the columns: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.

  Check that you can get all the way to identity, i.e. the system has a unique solution.

- $V_3$: No, $\vec{v}_2 + \vec{v}_3 = \vec{v}_1$, so the vectors are linearly dependent. Hence, they cannot form a basis for any vector space of any dimension.
3. Subspaces, Bases, and Dimension

For each of the sets $U_i \subseteq \mathbb{R}^3$ defined below, state whether it is a subspace or not. If it is a subspace, find a basis for it and state the dimension.

(a) $U_1 = \left\{ \begin{bmatrix} 2(x+y) \\ x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

**Answer:** $U_1$ is a subspace described by the basis $\mathcal{B}_1$, where

$\mathcal{B}_1 = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

The subspace has dimension 2, since there are 2 basis vectors.

(b) $U_2 = \left\{ \begin{bmatrix} x \\ y \\ z+1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$

**Answer:** $U_2$ is a subspace (in fact $\mathbb{R}^3$) of dimension 3, and a basis is the natural basis.

(c) $U_3 = \left\{ \begin{bmatrix} x \\ y \\ x+1 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

**Answer:** $U_3$ is not a subspace since it does not contain the zero vector and is therefore not closed under scalar multiplication.

(d) $U_4 = \left\{ \begin{bmatrix} x \\ y \\ (x+y)^2 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

**Answer:** $U_4$ is not a subspace, since it is not closed under scalar multiplication or vector addition and thus fails to satisfy the definition of a vector space. As an example, we can see that vector $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is in the space, but vector $\vec{v}_2 = 2\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ is not, so the scalar-multiplication property doesn’t hold; a similar argument can be made for vector addition.