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EECS 16A    Designing Information Devices and Systems I  
Spring 2020    Discussion 2B

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### 1. Matrix Multiplication

Consider the following matrices:

$$\mathbf{A}_1 = [1 \quad 4] \quad \mathbf{B}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix}$$

For each matrix multiplication problem, if the product exists, find the product by hand. Otherwise, explain why the product does not exist.

(a)  $\mathbf{A}_1\mathbf{B}_1$

**Answer:**  $\mathbf{A}_1$  is a  $1 \times 2$  vector and  $\mathbf{B}_1$  is a  $2 \times 1$  vector, so the product exists.  $\mathbf{A}_1\mathbf{B}_1 = 11$ .

(b)  $\mathbf{AB}$

**Answer:** Since both  $\mathbf{A}$  and  $\mathbf{B}$  are  $2 \times 2$  matrices, the product exists and is a  $2 \times 2$  matrix.

$$\mathbf{AB} = \begin{bmatrix} 11 & 6 \\ 12 & 7 \end{bmatrix}.$$

(c)  $\mathbf{BA}$

**Answer:** Since both  $\mathbf{A}$  and  $\mathbf{B}$  are  $2 \times 2$  matrices, the product exists and is a  $2 \times 2$  matrix.

$$\mathbf{BA} = \begin{bmatrix} 7 & 18 \\ 4 & 11 \end{bmatrix}.$$

(d)  $\mathbf{AC}$

**Answer:** Since  $\mathbf{A}$  is a  $2 \times 2$  matrix and  $\mathbf{C}$  is a  $2 \times 4$  matrix, the product exists and is a  $2 \times 4$  matrix.

$$\mathbf{AC} = \begin{bmatrix} 17 & 21 & 13 & 15 \\ 14 & 27 & 16 & 20 \end{bmatrix}.$$

(e)  $\mathbf{DC}$

**Answer:** Since  $\mathbf{C}$  is a  $2 \times 4$  matrix and  $\mathbf{D}$  is a  $4 \times 3$  matrix, the product does not exist. This is because the number of columns in the first matrix ( $\mathbf{D}$ ) should match the number of rows in the second matrix ( $\mathbf{C}$ ) for this product to be defined.

(f)  $\mathbf{CD}$  (Write down the dimensions of the product if it exists. For practice, you can compute the product on your own)

**Answer:** Since  $\mathbf{C}$  is a  $2 \times 4$  matrix and  $\mathbf{D}$  is a  $4 \times 3$  matrix, the product exists and is a  $2 \times 3$  matrix.

$$\mathbf{CD} = \begin{bmatrix} 100 & 33 & 75 \\ 52 & 29 & 56 \end{bmatrix}.$$

(g)  $\mathbf{EF}$  (Practice on your own)

**Answer:** Since  $\mathbf{E}$  and  $\mathbf{F}$  are both  $3 \times 3$  matrices, the product exists and is another  $3 \times 3$  matrix.

$$\mathbf{EF} = \begin{bmatrix} 53 & 50 & 64 \\ 34 & 70 & 57 \\ 33 & 90 & 44 \end{bmatrix}.$$

(h) **FE** (Practice on your own)

**Answer:** Since **E** and **F** are both  $3 \times 3$  matrices, the product exists and is another  $3 \times 3$  matrix.

$$\mathbf{FE} = \begin{bmatrix} 65 & 56 & 59 \\ 40 & 59 & 66 \\ 45 & 62 & 43 \end{bmatrix}.$$

## 2. Visualizing Matrices as Operations

This problem is going to help you visualize matrices as operations. For example, when we multiply a vector by a “rotation matrix,” we will see it “rotate” in the true sense here. Similarly, when we multiply a vector by a “reflection matrix,” we will see it be “reflected.” The way we will see this is by applying the operation to all the vertices of a polygon and seeing how the polygon changes.

Your TA will now show you how a unit square can be rotated, scaled, or reflected using matrices!

### Part 1: Rotation Matrices as Rotations

(a) We are given matrices  $\mathbf{T}_1$  and  $\mathbf{T}_2$ , and we are told that they will rotate the unit square by  $15^\circ$  and  $30^\circ$ , respectively. Design a procedure to rotate the unit square by  $45^\circ$  using only  $\mathbf{T}_1$  and  $\mathbf{T}_2$ , and plot the result in the IPython notebook. How would you rotate the square by  $60^\circ$ ?

**Answer:**

Apply  $\mathbf{T}_1$  and  $\mathbf{T}_2$  in succession to rotate the unit square by  $45^\circ$ . To rotate the square by  $60^\circ$ , you can either apply  $\mathbf{T}_2$  twice, or if you prefer variety, apply  $\mathbf{T}_1$  twice and  $\mathbf{T}_2$  once.

(b) Try to rotate the unit square by  $60^\circ$  using only one matrix. What does this matrix look like?

**Answer:** This matrix will look like the rotation matrix that rotates a vector by  $60^\circ$ . This matrix can be composed by multiplying  $\mathbf{T}_1$  by  $\mathbf{T}_1$  by  $\mathbf{T}_2$  (or equivalently,  $\mathbf{T}_2$  by  $\mathbf{T}_2$ ).

(c)  $\mathbf{T}_1$ ,  $\mathbf{T}_2$ , and the matrix you used in part (b) are called “rotation matrices.” They rotate any vector by an angle  $\theta$ . Show that a rotation matrix has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

where  $\theta$  is the angle of rotation. To do this consider rotating the unit vector  $\begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix}$  by  $\theta$  degrees using the matrix  $\mathbf{R}$ .

**(Definition:** A vector,  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix}$ , is a unit vector if  $\sqrt{v_1^2 + v_2^2 + \dots} = 1$ .)

(Hint: Use your trigonometric identities!)

**Answer:**

The reason the matrix is called a rotation matrix is because it transforms the unit vector  $\begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix}$  to

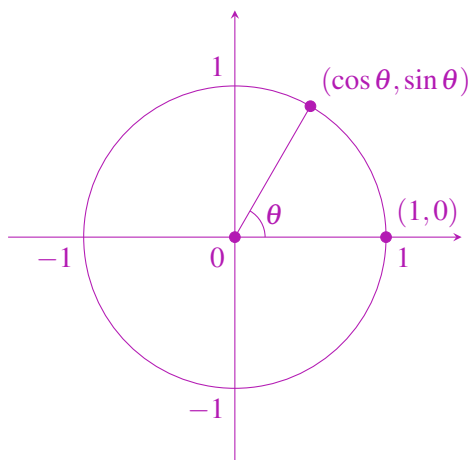
give  $\begin{bmatrix} \cos(\alpha + \theta) \\ \sin(\alpha + \theta) \end{bmatrix}$ .

Proof:

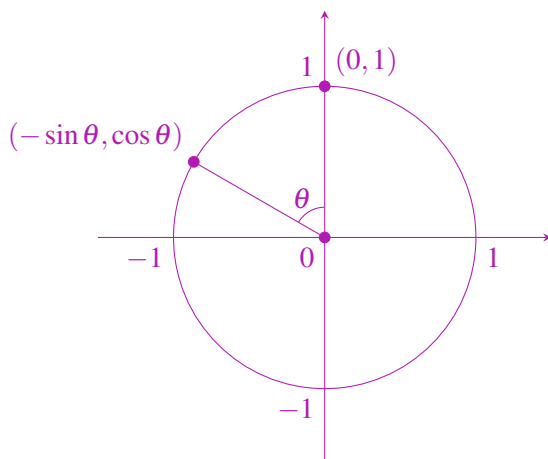
$$\begin{aligned} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix} &= \cos \alpha \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} + \sin \alpha \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos \alpha \cos \theta - \sin \alpha \sin \theta \\ \cos \alpha \sin \theta + \sin \alpha \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(\alpha + \theta) \\ \sin(\alpha + \theta) \end{bmatrix} \end{aligned}$$

**Alternative solution:**

Let's try to derive this matrix using trigonometry. Suppose we want to rotate the vector  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  by  $\theta$ .



We can use basic trigonometric relationships to see that  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  rotated by  $\theta$  becomes  $\begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$ . Similarly, rotating the vector  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  by  $\theta$  becomes  $\begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$ :



We can also scale these pre-rotated vectors to any length we want,  $\begin{bmatrix} x \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ y \end{bmatrix}$ , and we can observe graphically that they rotate to  $\begin{bmatrix} x \cos \theta \\ x \sin \theta \end{bmatrix}$  and  $\begin{bmatrix} -y \sin \theta \\ y \cos \theta \end{bmatrix}$ , respectively. Rotating a vector solely in the

$x$ -direction produces a vector with both  $x$  and  $y$  components, and, likewise, rotating a vector solely in the  $y$ -direction produces a vector with both  $x$  and  $y$  components.

Finally, if we want to rotate an arbitrary vector  $\begin{bmatrix} x \\ y \end{bmatrix}$ , we can combine what we derived above. Let  $x'$  and  $y'$  be the  $x$  and  $y$  components after rotation.  $x'$  has contributions from both  $x$  and  $y$ :  $x' = x \cos \theta - y \sin \theta$ . Similarly,  $y'$  has contributions from both components as well:  $y' = x \sin \theta + y \cos \theta$ . Expressing this in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, we've derived the 2-dimensional rotation matrix.

- (d) Now, we want to get back the original unit square from the rotated square in part (b). What matrix should we use to do this? (**Note:** Don't use inverses! Answer this question using your intuition, we will visit inverses very soon in lecture!)

**Answer:**

Use a rotation matrix that rotates by  $-60^\circ$ .

$$\begin{bmatrix} \cos(-60^\circ) & -\sin(-60^\circ) \\ \sin(-60^\circ) & \cos(-60^\circ) \end{bmatrix}$$

- (e) Use part (d) to obtain the “inverse” rotation matrix for a matrix that rotates a vector by  $\theta$ . Multiply the inverse rotation matrix with the rotation matrix and vice-versa. What do you get?

**Answer:**

The inverse matrix is as follows:

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

We can see that for any  $\vec{v} \in \mathbb{R}^2$  that the product of the rotation matrix with  $\vec{v}$  followed by the product of the inverse results in the original  $\vec{v}$ .

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \left( \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \vec{v} \right) = \vec{v}$$

- (f) What are the matrices that reflect a vector about the (i)  $x$ -axis, (ii)  $y$ -axis, and (iii)  $x = y$

**Answer:**

The matrix that reflects about the  $x$ -axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The matrix that reflects about the  $y$ -axis:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the matrix that reflects about  $x = y$ :

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

## Part 2: Commutativity of Operations

A natural question to ask is the following: Does the *order* in which you apply these operations matter? Follow your TA to obtain the answers to the following questions!

- (a) Let's see what happens to the unit square when we rotate the square by  $60^\circ$  and then reflect it along the  $y$ -axis.
- (b) Now, let's see what happens to the unit square when we first reflect the square along the  $y$ -axis and then rotate it by  $60^\circ$ .

**Answer:** (For parts (a) and (b)): The two operations are not the same.

- (c) Try to do steps (a) and (b) by multiplying the reflection and rotation matrices together (in the correct order for each case). What does this tell you?

**Answer:**

The resulting matrices that are obtained (by multiplying the two matrices) are different depending on the order of multiplication.

- (d) If you reflected the unit square twice (along any pair of axes), do you think the order in which you applied the reflections would matter? Why/why not?

**Answer:**

It turns out that reflections are not commutative unless the two reflection axes are perpendicular to each other. For example, if you reflect about the  $x$ -axis and the  $y$ -axis, it is commutative. But if you reflect about the  $x$ -axis and  $x = y$ , it is not commutative.

## Part 3: Distributivity of Operations

- (a) The distributivity property of matrix-vector multiplication holds for any vectors and matrices. Show for general  $\mathbf{A} \in \mathbb{R}^{2 \times 2}$  and  $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$  that  $\mathbf{A}(\vec{v}_1 + \vec{v}_2) = \mathbf{A}\vec{v}_1 + \mathbf{A}\vec{v}_2$ .

**Answer:** Matrix-vector multiplication distributes because scalar multiplication distributes.

$$\mathbf{A}(\vec{v}_1 + \vec{v}_2) = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} (\vec{v}_1 + \vec{v}_2) \quad (1)$$

$$= (v_{11} + v_{21})\vec{a}_1 + (v_{12} + v_{22})\vec{a}_2 \quad (2)$$

$$= \begin{bmatrix} a_{11}(v_{11} + v_{21}) + a_{12}(v_{12} + v_{22}) \\ a_{21}(v_{11} + v_{21}) + a_{22}(v_{12} + v_{22}) \end{bmatrix} \quad (3)$$

$$= \begin{bmatrix} a_{11}v_{11} + a_{12}v_{12} \\ a_{21}v_{11} + a_{22}v_{12} \end{bmatrix} + \begin{bmatrix} a_{11}v_{21} + a_{12}v_{22} \\ a_{21}v_{21} + a_{22}v_{22} \end{bmatrix} \quad (4)$$

$$= \mathbf{A}\vec{v}_1 + \mathbf{A}\vec{v}_2 \quad (5)$$