EECS 16A Designing Information Devices and Systems I
Spring 2020 Discussion 2A

1. Computations: Inner product and matrix-vector multiplication

(a) For each of the following pairs of vectors, compute their inner product and determine whether they are orthogonal.

i.

$$\vec{a} = \begin{bmatrix} 1 \\ 6 \\ 11 \end{bmatrix} \ \vec{b} = \begin{bmatrix} -6 \\ 1 \\ 2 \end{bmatrix}$$

ii.

$$\vec{a} = \begin{bmatrix} 2 \\ 6 \\ 12 \\ -4 \end{bmatrix} \ \vec{b} = \begin{bmatrix} 6 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

Answer:

i. $\vec{a}^T\vec{b} = (1)(-6) + (6)(1) + (11)(2) = 22$. Since this is not zero, a and b are not orthogonal.

ii. $\vec{a}^T\vec{b} = (2)(6) + (6)(2) + (12)(-1) + (-4)(3) = 0$. They are orthogonal.

(b) Perform matrix vector multiplication to compute $A\vec{b}$ in each of the following cases:

i.

$$A = \begin{bmatrix} 1 & 6 \\ 2 & -7 \end{bmatrix} \ \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

ii.

$$A = \begin{bmatrix} 1 & 9 & 2 \\ 7 & 10 & -7 \\ -1 & 2 & -8 \end{bmatrix} \ \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Answer:

i. Let $a_i \in \mathbb{R}^3$ represent the transpose of the i_{th} row of the A matrix. To find the i_{th} entry of the $A\vec{b}$ vector we find the inner product of a_i and b. In this case, we get

$$A\vec{b} = \begin{bmatrix} 13 \\ -12 \end{bmatrix}$$

ii. Let $A_i \in \mathbb{R}^3$ represent the i_{th} column of A and let $b_i \in \mathbb{R}$ represent the i_{th} component of b.

$$A\vec{b} = A_1 b_1 + A_2 b_2 + A_3 b_3 = 1 \times \begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix} + 3 \times \begin{bmatrix} 2 \\ -7 \\ -8 \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \\ -25 \end{bmatrix}$$

2. Span basics

(a) What is $\text{span}\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right\}$?

Answer: $\text{span}\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right\}$ contains any vector $\vec{v}$ that can be written as

$$\vec{v} = \alpha_1 \begin{bmatrix}1\\2\\0\end{bmatrix} + \alpha_2 \begin{bmatrix}2\\1\\0\end{bmatrix}$$

We realize that any vector whose last component is 0 can be written in this form and any vector whose last component is nonzero cannot. Hence, the required span is the set of all vectors that can be written in the form $\begin{bmatrix}*\\*\\0\end{bmatrix}$.

(b) Is $\begin{bmatrix}5\\5\\0\end{bmatrix}$ in $\text{span}\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right\}$?

Answer: Yes. We realize from inspection that

$$\begin{bmatrix}5\\5\\0\end{bmatrix} = \frac{5}{3}\begin{bmatrix}1\\2\\0\end{bmatrix} + \frac{5}{3}\begin{bmatrix}2\\1\\0\end{bmatrix}$$

(c) What is a possible choice for $\vec{v}$ that would make $\text{span}\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \vec{v}\right\} = \mathbb{R}^3$?

Answer: From part (a), we realize that any vector whose last component is 0 can be written as a linear combination of the two vectors already in the set. Hence, if we include, for example, $\begin{bmatrix}0\\0\\1\end{bmatrix}$ into the set, then we should be able to reach any vector in $\mathbb{R}^3$. Any vector whose last component is non-zero is a valid addition to the set to achieve the desired span.

(d) For what values of b_1, b_2, b_3 is the following system of linear equations consistent?

$$\begin{bmatrix}1 & 2\\2 & 1\\0 & 0\end{bmatrix}\vec{x} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

Answer: For the system of linear equations to be consistent, there must exist some x such that the equality above holds. Performing matrix vector multiplication, we can rewrite the above equality as

$$x_1\begin{bmatrix}1\\2\\0\end{bmatrix} + x_2\begin{bmatrix}2\\1\\0\end{bmatrix} = \vec{b}$$

The question now becomes: which vectors $\vec{b}$ can be written in the above form i.e as a linear combination of the columns of A? This is exactly the definition of span, and the answer must be the same as that from part (a).

3. Span Proofs

Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, show the following:

(a)

$$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}, \text{ where } \alpha \text{ is a non-zero scalar}$$

In other words, we can scale our spanning vectors and not change their span.

(b)

$$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \text{span}\{\vec{v}_2, \vec{v}_1, \dots, \vec{v}_n\}$$

In other words, we can swap the order of our spanning vectors and not change their span.

Answer:

(a) Suppose we have some arbitrary $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. For some scalars a_i:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \left(\frac{a_1}{\alpha}\right)\alpha\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n.$$

Scalar multiplication cancels out. Thus, we have shown that $\vec{q} \in \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. Therefore, we have $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subseteq \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. Now, we must show the other direction. Suppose we have some arbitrary $\vec{r} \in \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. For some scalars b_i:

$$\vec{r} = b_1(\alpha\vec{v}_1) + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n = (b_1\alpha)\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n.$$

Thus, we have shown that $\vec{r} \in \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. Therefore, we now have $\text{span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. Combining this with the earlier result, the spans are thus the same.

(b) Suppose $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. For some scalars a_i:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = a_2\vec{v}_2 + a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$$

Swapping the order in addition does not affect the sum, so $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_2, \vec{v}_1, \dots, \vec{v}_n\}$. Similarly, starting with some $\vec{r} \in \text{span}\{\vec{v}_2, \vec{v}_1, \dots, \vec{v}_n\}$, again swapping the order does not affect the sum, so putting both together, the spans are thus the same.