EECS 16A Designing Information Devices and Systems I
Spring 2020 Discussion 1B

1. Linear or Nonlinear

Determine whether the following functions (f: $\mathbb{R}^2 \to \mathbb{R}$) are linear or nonlinear.

(a)

$$f(x_1, x_2) = 3x_1 + 4x_2$$

Answer: To check for linearity, check for superposition (additivity) and homogeneity (multiplicative scaling).
In other words we must check that:

$$f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) = \alpha f(x_1, x_2) + \beta f(y_1, y_2) \quad \forall \alpha, \beta, x_1, x_2, y_1, y_2 \in \mathbb{R}$$

Linear

$$\begin{aligned} f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) &= 3(\alpha x_1 + \beta y_1) + 4(\alpha x_2 + \beta y_2) \\ &= \alpha(3x_1 + 4x_2) + \beta(3y_1 + 4y_2) \\ &= \alpha f(x_1, x_2) + \beta f(y_1, y_2) \end{aligned}$$

Alternatively you can state that this function is linear because it is of the form:

$$f(x_1, x_2) = a_1 x_1 + a_2 x_2$$

where a_1 and a_2 are constants.

(b)

$$f(x_1, x_2) = e^{x_2} + x_1^2$$

Answer: To check for linearity, check for additivity and homogeneity (multiplicative scaling).
In other words we must check that:

$$f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) = \alpha f(x_1, x_2) + \beta f(y_1, y_2) \quad \forall \alpha, \beta, x_1, x_2, y_1, y_2 \in \mathbb{R}$$

Nonlinear

$$\begin{aligned} f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) &= e^{\alpha x_2 + \beta y_2} + (\alpha x_1 + \beta y_1)^2 \\ &\neq e^{\alpha x_2} + (\alpha x_1)^2 + e^{\beta y_2} + (\beta y_1)^2 \\ &= \alpha f(x_1, x_2) + \beta f(y_1, y_2) \end{aligned}$$

Alternatively you can state that this function is nonlinear because it is NOT of the form:

$$f(x_1, x_2) = a_1 x_1 + a_2 x_2$$

where a_1 and a_2 are constants.

(c)

$$f(x_1, x_2) = x_2 - x_1 + 3$$

Answer: To check for linearity, check for additivity and homogeneity (multiplicative scaling). In other words we must check that:

$$f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) = \alpha f(x_1, x_2) + \beta f(y_1, y_2) \quad \forall \alpha, \beta, x_1, x_2, y_1, y_2 \in \mathbb{R}$$

Nonlinear, in fact this function is affine (see notes for more details).
You may simply state that this function doesn't satisfy homogeneity when scaled by 0.
Alternatively you can show:

$$\begin{aligned} f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) &= (\alpha x_2 + \beta y_2) - (\alpha x_1 + \beta y_1) + 3 \\ &\neq \alpha(x_2 - x_1 + 3) + \beta(y_2 - y_1 + 3) \\ &= \alpha f(x_1, x_2) + \beta f(y_1, y_2) \end{aligned}$$

Alternatively you can state that this function is nonlinear because it is NOT of the form:

$$f(x_1, x_2) = a_1 x_1 + a_2 x_2$$

where a_1 and a_2 are constants.

2. Solving Systems of Equations

(a) Systems of linear equations can either have one solution, an infinite number of solutions, or no solution at all. For the following systems of equations, state whether there is a unique solution, no solution, or an infinite number of solutions. If there are an infinite number of solutions give one possible solution.

$$\text{i.} \begin{cases} 49x + 7y = 49 \\ 42x + 6y = 42 \end{cases}$$

Answer:

$$\left[\begin{array}{cc|c} 49 & 7 & 49 \\ 42 & 6 & 42 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{1}{7} & 1 \\ 42 & 6 & 42 \end{array}\right] \qquad \text{using } R_1 \leftarrow \tfrac{1}{49} R_1$$

$$\to \left[\begin{array}{cc|c} 1 & \frac{1}{7} & 1 \\ 0 & 0 & 0 \end{array}\right] \qquad \text{using } R_2 \leftarrow R_2 - 42 R_1$$

Let x be a free variable, such as a. Set x = a, solve for y in terms of a.

$$\begin{aligned} x &= a \\ a + \frac{1}{7} y &= 1 \\ y = 7(1-a) &= 7 - 7a \end{aligned}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ 7 - 7a \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \end{bmatrix} + \begin{bmatrix} a \\ -7a \end{bmatrix}, \forall a \in \mathbb{R}$$

ii. $\begin{cases} 5x + 3y = -21 \\ 2x + y = -9 \end{cases}$

Answer:

$$\left[\begin{array}{cc|c} 5 & 3 & -21 \\ 2 & 1 & -9 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{3}{5} & -\frac{21}{5} \\ 2 & 1 & -9 \end{array}\right] \quad \text{using } R_1 \leftarrow \tfrac{1}{5}R_1$$

$$\to \left[\begin{array}{cc|c} 1 & \frac{3}{5} & -\frac{21}{5} \\ 0 & -\frac{1}{5} & -\frac{3}{5} \end{array}\right] \quad \text{using } R_2 \leftarrow R_2 - 2R_1$$

$$\to \left[\begin{array}{cc|c} 1 & \frac{3}{5} & -\frac{21}{5} \\ 0 & 1 & 3 \end{array}\right] \quad \text{using } R_2 \leftarrow -5R_2$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & -6 \\ 0 & 1 & 3 \end{array}\right] \quad \text{using } R_1 \leftarrow R_1 - -\tfrac{3}{5}R_2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$$

iii. $\begin{cases} 49x + 7y = 60 \\ 42x + 6y = 30 \end{cases}$

Answer:

$$\left[\begin{array}{cc|c} 49 & 7 & 60 \\ 42 & 6 & 30 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{1}{7} & \frac{60}{49} \\ 42 & 6 & 30 \end{array}\right] \quad \text{using } R_1 \leftarrow \tfrac{1}{49}R_1$$

$$\to \left[\begin{array}{cc|c} 1 & \frac{1}{7} & \frac{60}{49} \\ 0 & 0 & \frac{5}{7} \end{array}\right] \quad \text{using } R_2 \leftarrow R_2 - 42R_1$$

No solution

iv. $\begin{cases} 2x + 2y + 4z = -1 \\ y + z = -2 \\ x + 2y + 3z = 2 \end{cases}$

Answer:

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & -1 \\ 0 & 1 & 1 & -2 \\ 1 & 2 & 3 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & -\frac{1}{2} \\ 0 & 1 & 1 & -2 \\ 1 & 2 & 3 & 2 \end{array}\right] \quad \text{using } R_1 \leftarrow -\tfrac{1}{2}R_1$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & -\frac{1}{2} \\ 0 & 1 & 1 & -2 \\ 0 & 1 & 1 & \frac{5}{2} \end{array}\right] \quad \text{using } R_3 \leftarrow R_3 - R_1$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & -\frac{1}{2} \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & \frac{9}{2} \end{array}\right] \quad \text{using } R_3 \leftarrow R_3 - R_2$$

No solution

$$\text{v.} \begin{cases} 2x + 2y + 4z = 6 \\ \quad\quad\;\; y + z = 1 \\ x + 2y + 3z = 4 \end{cases}$$

Answer:

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 6 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{array}\right] \quad \text{using } R_1 \leftarrow \tfrac{1}{2}R_1$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \quad \text{using } R_3 \leftarrow R_3 - R_1$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{using } R_3 \leftarrow R_3 - R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{using } R_1 \leftarrow R_1 - R_2$$

Let x be a free variable, such as a. Set x = a, solve for y and z in terms of a.

$$x = a$$
$$a + z = 2$$
$$z = 2 - a$$
$$y + (2 - a) = 1$$
$$y = -1 + a$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ -1+a \\ 2-a \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} a \\ a \\ -a \end{bmatrix}, \forall a \in \mathbb{R}$$

$$\text{vi.} \begin{cases} x + y + z = 4 \\ 3z = 6 \\ y + z = 3 \end{cases}$$

Answer:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 0 & 3 & 6 \\ 0 & 1 & 1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 3 & 6 \end{array}\right] \quad \text{swapping } R_2, R_3$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \text{using } R_3 \leftarrow \tfrac{1}{3}R_3$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \text{using } R_1 \leftarrow R_1 - R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \text{using } R_2 \leftarrow R_2 - R_3$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

(b) Systems of equations can also be interpreted graphically. We will try to build a graphical intuition for the results you found in the previous part. Follow along as your TA walks through `dis1B.ipynb`.

Answer:

i. The lines lie on top of one another (i.e. they are the same line), so there are an infinite number of solutions. This system is referred to as underdetermined, which means that there are more unknowns than equations. Though it appears we have two equations and two unknowns, dividing the top equation by 7 and the bottom one by 6 quickly reveals that they are both the same equation.

ii. The lines intersect at one point, so the solution is unique.

iii. The lines do not intersect (a little algebraic manipulation on the equations would reveal that the two lines are parallel). There is no solution.

iv. The lines do not intersect. There is no solution.

v. The intersection of the planes is a line, so there is an infinite number of solutions. This system is also underdetermined. Subtracting the second equation solution from the third and multiplying the result by two yields the first equation. In other words including the first equation is redundant because any point that satisfies the second and third equation will certainly satisfy the first. In effect, we have two equations and one unknown.

vi. The intersection of the planes is a single point, so there is a unique solution.

3. Vectors Introduction to vectors and vector addition.

Definitions:

Vector: An ordered list of elements - for example:

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

$\mathbb{R}$ or $\mathbb{R}^1$: The set of all real numbers (i.e. the real line)

$\mathbb{R}^2$: The set of all two-element vectors with real numbered entries (i.e. plane of 2×1 vectors) - for example:

$$\vec{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \in \mathbb{R}^2$$

$\mathbb{R}^3$: The set of all three-element vectors with real numbered entries (i.e. 3-space of 3×1 vectors) - for example:

$$\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \in \mathbb{R}^3$$

$\mathbb{R}^n$: The set of all n-element vectors with real numbered entries (i.e. n-space of $n \times 1$ vectors)

(a) Are the following vectors in $\mathbb{R}^2$?

i. $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$

ii. $\begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}$

Answer:

i. Yes, it is a two element vector of real numbered entries.

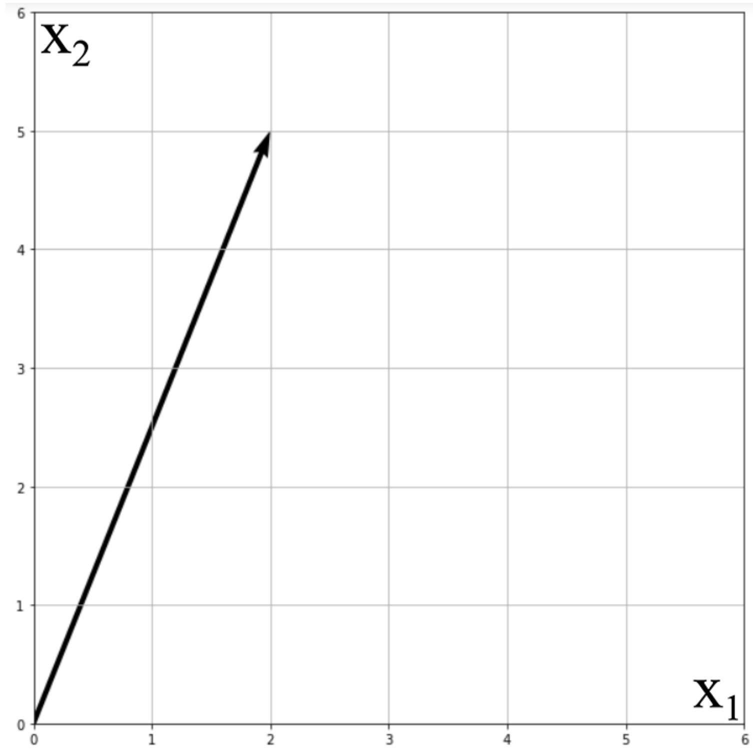
ii. No, it is a three element vector of real numbered entries.

(b) Graphically show the vectors:

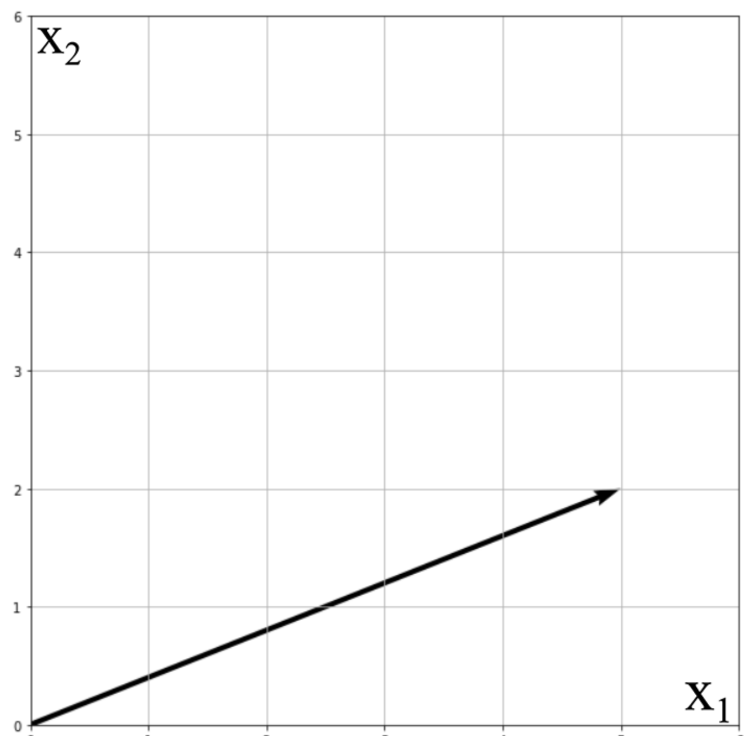
i. $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

ii. $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

Answer:



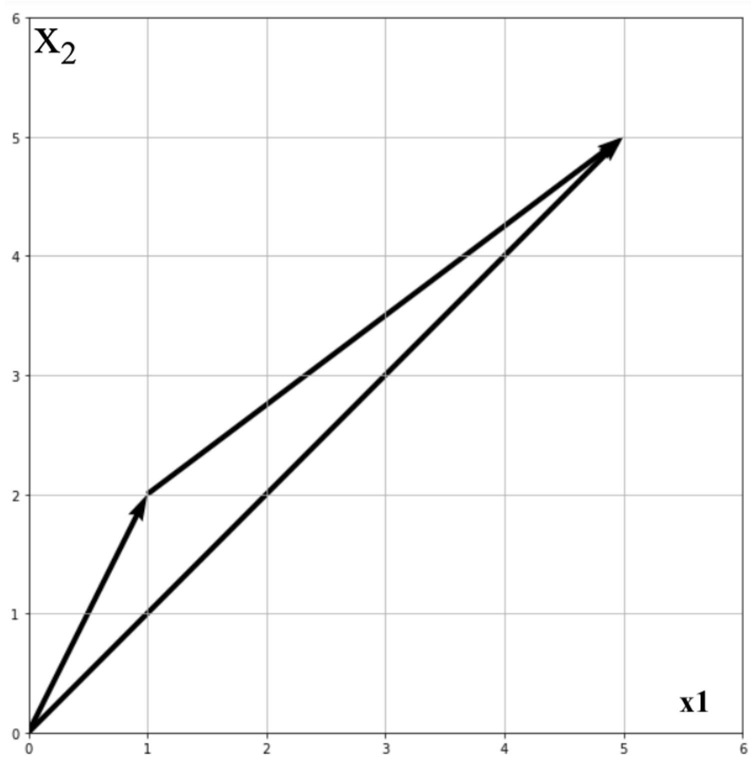
i.



ii.

(c) Graphically show the vector sum and check your answer algebraically:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$



Answer:

$$\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}5\\5\end{bmatrix}$$