**Eigen-Stuff, Convergence, and Steady State**

Remember the definition of eigenvectors and eigenvalues:

\[ A\vec{v} = \lambda \vec{v} \]

where \( \vec{v} \) is an eigenvector and \( \lambda \) is an eigenvalue. For the purposes of steady state/convergence analysis, we think of \( A \) as an operator which acts on some state vector \( \vec{v} \) at each time step. All behavior can be generalized down to a few cases:

- **\( \lambda = 1 \):** Any vector in the eigenspace associated with \( \lambda = 1 \) \( v_{\lambda=1} \) is a “steady state”. In other words,
  \[ A v_{\lambda=1} = v_{\lambda=1} \]
  So if we keep applying \( A \) to \( v_{\lambda=1} \), the state doesn’t change.

- **\( \lambda = -1 \):** Say it has an eigenspace containing \( v_{\lambda=-1} \). Let’s call some vector in that eigenspace \( v_{\lambda=-1} \). If we keep applying \( A \) to this vector, it just bounces back and forth between the positive and negative versions of the vector:
  \[ \lim_{n \to \infty} (A^n v_{\lambda=-1}) = \lim_{n \to \infty} (-1)^n v_{\lambda=-1} \]
  The limit doesn’t exist! It just goes back and forth forever and ever, meaning if you start out the system in state \( v_{\lambda=-1} \), it will never converge. This shows up in controls when people talk about “marginal instability” and in circuits when people talk about “ringing”.

- **\( |\lambda| < 1 \):** We’ll say this eigenvalue has an eigenspace containing \( v_{<1} \). Repeating the process from above:
  \[ \lim_{n \to \infty} (A^n v_{<1}) = \lim_{n \to \infty} \lambda^n v_{<1} \]
  Because \( |\lambda| < 1 \), multiplying it by itself repeatedly just makes the magnitude smaller and smaller. This holds whether \( \lambda \) is positive or negative, and what this means is that if the system starts out in \( v_{<1} \), after an infinite amount of time the state will approach \( \vec{0} \). You can think of it as asymptotically approaching zero. For physical systems, (not in scope of the class) this is related to damping (overdamped being \( \lambda > 0 \) and underdamped being \( \lambda < 0 \)).

- **\( |\lambda| > 1 \):** Now we’ll say this eigenvalue has an eigenspace containing \( v_{>1} \).
  \[ \lim_{n \to \infty} (A^n v_{>1}) = \lim_{n \to \infty} \lambda^n v_{>1} \]
  In this case, the state vector just keeps increasing in size to infinity, meaning if you start out the system in state \( v_{>1} \), your system will never converge to a steady state.
The question then arises: what happens if we have a state vector $\vec{s}$ which is not an eigenvector? We can think of $\vec{s}$ as a linear combination of the $N$ eigenvectors:

$$A\vec{v}_i = \lambda_i \vec{v}_i$$

$$\vec{s} = \sum_{i=1}^{N} c_i \vec{v}_i$$

After many many time steps:

$$\lim_{n \to \infty} (A^n\vec{s}) = \lim_{n \to \infty} \left( A^n \sum_{i=1}^{N} c_i \vec{v}_i \right)$$

$$= \lim_{n \to \infty} \left( \sum_{i=1}^{N} c_i \lambda_i^n \vec{v}_i \right)$$

Now what we care about is the coefficients $c_i$ as it applies to the cases we described above. We’ve now broken our state $\vec{s}$ into its eigenvector components, and we know how those components will behave when acted upon by $A$ and thus, what $\vec{s}$ will look like after applying $A$ many times.

1. A Tropical Tale of Triumph: Does Pineapple Come Out on Top? (Fall 2018 Midterm 1)

   (Based on a true story) During a discussion section, one of your TAs, Nick, makes the claim that pineapple belongs on pizza. Another TA, Elena, strongly disagrees. Naturally, a war starts and students begin to flock to the TA they agree with, switching discussion sections every week. Some students don’t have an opinion and go to Lydia’s section since she is neutral in the matter. As a 16A student, you want to analyze this war to see how it will play out.

   (a) You manage to capture the behavior of the students as a transition matrix, but want to visualize it. You’ve written out the transition matrix $M$:

$$M = \begin{bmatrix}
0.5 & 0 & 0 \\
0.5 & 0 & 0 \\
0.5 & 0 & 0 \\
0.25 & 0.5 & 1 \\
0.25 & 0.5 & 1 \\
\end{bmatrix}$$

   such that

$$\begin{bmatrix}
x_{Elena}[n+1] \\
x_{Nick}[n+1] \\
x_{Lydia}[n+1]
\end{bmatrix} = M \begin{bmatrix}
x_{Elena}[n] \\
x_{Nick}[n] \\
x_{Lydia}[n]
\end{bmatrix}.$$ 

Each element of the state vector $\vec{x}[n] = \begin{bmatrix} x_{Elena}[n] & x_{Nick}[n] & x_{Lydia}[n] \end{bmatrix}^T$ represents the number of students attending that section at timestep $n$. **Fill in values in the boxes in Figure 1 below** such that the diagram represents the transition matrix $M$. 

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Figure 1: Flow diagram for discussion sections

**Answer:** Writing out the system of equations,

\[
\begin{align*}
\chi_{\text{Elena}}[t+1] &= \frac{1}{2} \chi_{\text{Elena}}[t] \\
\chi_{\text{Nick}}[t+1] &= \frac{1}{4} \chi_{\text{Elena}}[t] + \frac{1}{2} \chi_{\text{Nick}}[t] + \frac{1}{4} \chi_{\text{Lydia}}[t] \\
\chi_{\text{Lydia}}[t+1] &= \frac{1}{4} \chi_{\text{Elena}}[t] + \frac{1}{2} \chi_{\text{Nick}}[t]
\end{align*}
\]

It helps to write these out so you don’t accidentally use the transpose!

Filling in the diagram from above:
(b) Your friend Vlad tells you that your transition matrix $\mathbf{M}$ was wrong, and gives you a new transition matrix $\mathbf{S}$, which has a steady state. In order to find who wins the war, you need to find how many students end up in each section after everything has settled. Find a vector $\mathbf{x}$ that represents a steady state of $\mathbf{S}$.

$$
\mathbf{S} = \begin{bmatrix}
0.2 & 0.5 & 0 \\
0.5 & 0.5 & 0 \\
0.3 & 0 & 1 \\
\end{bmatrix}
$$

**Answer:** To find the eigenvector corresponding to $\lambda = 1$, use the equation relating eigenvalues to eigenvectors: $\mathbf{S}\mathbf{x} = \lambda\mathbf{x}$ and substitute in 1 for $\lambda$.

$$
\mathbf{S}\mathbf{x} = \lambda\mathbf{x}
= \mathbf{x} \\
\iff \lambda = 1
\iff \mathbf{S}\mathbf{x} - \mathbf{x} = \mathbf{0}
\iff (\mathbf{S} - I)\mathbf{x} = \mathbf{0}
$$

To find the eigenvector, we must solve for the vector $\mathbf{x}$ which satisfies the above equation. In other words, we need to find Null($\mathbf{S} - I$).

$$
\mathbf{S} - I = \begin{bmatrix}
0.2 & 0.5 & 0 \\
0.5 & 0.5 & 0 \\
0.3 & 0 & 1 \\
\end{bmatrix} - \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
-0.8 & 0.5 & 0 \\
0.5 & -0.5 & 0 \\
0.3 & 0 & 0 \\
\end{bmatrix}
$$
We do this using Gaussian elimination

\[
\begin{bmatrix}
-0.8 & 0.5 & 0 \\
0.5 & -0.5 & 0 \\
0.3 & 0 & 0
\end{bmatrix}
\xrightarrow{R_1 + R_3 \mapsto R_1}
\begin{bmatrix}
-0.5 & 0.5 & 0 \\
0.5 & -0.5 & 0 \\
0.3 & 0 & 0
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
0.3 & 0 & 0 \\
0.5 & -0.5 & 0 \\
-0.5 & 0.5 & 0
\end{bmatrix}
\]

Rearranging the equations from our upper triangular matrix above, we get the following:

\[
\begin{align*}
x_1 & = 0 \\
x_2 & = 0
\end{align*}
\]

Setting \(x_3\) as a free variable, we find

\[
\text{Null}(S - I) = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

where the eigenvector \(\bar{x}\) is any vector contained within this space. During the exam, the clarification was added to ask for a nonzero vector. Any nonzero scalar multiple of \(\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T\) is a correct solution.

(c) Your other friend Gireeja points out that the arguments are causing new people to join the sections and others to leave entirely. In other words, the system is not conservative! The new system can be modeled with a state transition matrix \(A\) that has the following eigenvalue/eigenvector pairings:

\[
\begin{align*}
\lambda_1 & = 1 : \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
\lambda_2 & = \frac{1}{2} : \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
\lambda_3 & = 2 : \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\end{align*}
\]

You want the number of students in sections to stabilize. Which of the vectors below represent \textbf{steady states} of the system, i.e. \(\vec{x}\) such that \(A\vec{x} = \vec{x}\)? \textbf{Fill in the circle(s) to the left of these vector(s)}.
Answer: The state $\vec{x}$ can be written as a linear combination of the eigenvectors:

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$$

Applying $A$ to $\vec{x}$,

$$A \vec{x} = c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 + c_3 \lambda_3 \vec{v}_3$$

$$\vec{x} = c_1 \vec{v}_1 + \frac{1}{2} c_2 \vec{v}_2 + 2c_3 \vec{v}_3 \iff A \vec{x} = \vec{x}$$

$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = c_1 \vec{v}_1 + \frac{1}{2} c_2 \vec{v}_2 + 2c_3 \vec{v}_3$

Given the condition that we need a steady state and because $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are linearly independent, we know that

• $c_2 = c_3 = 0$
• $c_1 \geq 0$. If $c_1 = 0$, this means there are no students to begin with, which makes for a very boring (but extremely stable!) war. If $c_1 > 0$, it just satisfies the eigenvalue equation $A c_1 \vec{v}_1 = c_1 \vec{v}_1$.

Long story short, we’re looking for scaled versions of $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(d) Assume we are still working with the same state transition matrix $A$ as in part (c). Which of the vectors below represent initial states such that the number of students in the sections keeps growing? Fill in the circle(s) to the left of these vector(s).

$$\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 513 \\ 513 \\ 0 \end{bmatrix}, \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1026 \\ 0 \end{bmatrix}, \begin{bmatrix} 1026 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 513 \\ 0 \\ 12 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1026 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
Answer: Writing \( \vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 \), we want to see what situations down the line will never lead to a steady state.

\[
A^n \vec{x} = \lambda_1^n c_1 \vec{v}_1 + \lambda_2^n c_2 \vec{v}_2 + \lambda_3^n c_3 \vec{v}_3
\]

\[
= c_1 \vec{v}_1 + \frac{1}{2^n} c_2 \vec{v}_2 + 2^n c_3 \vec{v}_3
\]

After waiting an infinite amount of time, i.e. \( n \to \infty \), the \( \vec{v}_1 \) component will stay steady; the \( \vec{v}_2 \) component will go to zero; and the \( \vec{v}_3 \) component will explode to infinity unless \( c_3 = 0 \). Thus, in order for things to explode, the following conditions must be true:

- \( c_3 > 0 \)
- \( c_2 \) and \( c_1 \) can be any nonnegative (including zero) value

In other words, we want vectors which have a nonzero \( \vec{v}_3 \) component. This causes the system to never settle into a steady state, i.e. there will always be people being drawn into the flame war of pineapples on pizza.

\[
\begin{bmatrix}
5 \\
0 \\
0
\end{bmatrix} \quad \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} \quad \begin{bmatrix}
513 \\
513 \\
0
\end{bmatrix} \quad \begin{bmatrix}
0 \\
12 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix} \quad \begin{bmatrix}
1026 \\
0 \\
0
\end{bmatrix} \quad \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} \quad \begin{bmatrix}
0 \\
1026 \\
0
\end{bmatrix}
\]

Note: Had there been an eigenvalue of -1 with an associated eigenvector \( \vec{v}_4 \), applying \( A \) to \( \vec{v}_4 \) repeatedly would just lead to oscillation between the positive and negative vectors. This is another form of non-convergence.

(e) Again assume we are still working with the same state transition matrix \( A \) as in part (c). Which of the vectors below represent initial states such that everyone leaves the system, i.e. \( \lim_{n \to \infty} A^n \vec{x} = \vec{0} \)? Fill in the circle(s) to the left of these vector(s).

\[
\begin{bmatrix}
5 \\
0 \\
0
\end{bmatrix} \quad \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} \quad \begin{bmatrix}
513 \\
513 \\
0
\end{bmatrix} \quad \begin{bmatrix}
0 \\
12 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix} \quad \begin{bmatrix}
1026 \\
0 \\
0
\end{bmatrix} \quad \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} \quad \begin{bmatrix}
0 \\
1026 \\
0
\end{bmatrix}
\]

Answer: Defining \( \vec{x} \) and finding \( A\vec{x} \):

\[
\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3
\]

\[
A^n \vec{x} = \lambda_1^n c_1 \vec{v}_1 + \lambda_2^n c_2 \vec{v}_2 + \lambda_3^n c_3 \vec{v}_3
\]

After waiting an infinite amount of time, i.e. \( n \to \infty \), the \( \vec{v}_1 \) component will stay steady; the \( \vec{v}_2 \) component will go to zero; and the \( \vec{v}_3 \) component will explode to infinity unless \( c_3 = 0 \).
• $c_1 = 0$ so there are no nonzero steady states.
• $c_3 = 0$ so the system does not explode.
• $c_2 \geq 0$

We know that after a long time, any state that is a scaled form of $\vec{v}_2$ associated with eigenvalue $\lambda_2 = \frac{1}{2}$ will approach $\vec{0}$, i.e. the number of people will keep halving until there are none left.

tl;dr, we’re looking for any scaled version of $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(5) Let us generalize the idea of convergence. Consider the following system:

$$\vec{x}[n + 1] = T\vec{x}[n]$$

where $\vec{x}$ is a vector with $N$ elements and $T$ is any $N \times N$ matrix unrelated to the previous parts. $T$ has $N$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$, and $N$ associated eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N$ such that $T\vec{v}_i = \lambda_i \vec{v}_i$ for $1 \leq i \leq N$. Let $|\lambda_1| > 1$. Prove that there exists at least one initial state $\vec{x}[0]$ for this system such that it does not converge to a steady state.

**Answer:** We’ll generalize in terms of $i$, and then set $i = 1$. Let $\vec{v}_i$ be a nonzero eigenvector corresponding to the eigenvalue $\lambda_i$. We know that such a vector must exist because $\det(T - \lambda_i I) = 0$. Also, let $\vec{x}[0] = \vec{v}_i$, and consider $\lim_{n \to \infty} T^n \vec{x}[0]$.

$$\lim_{n \to \infty} T^n \vec{x}[0] = \lim_{n \to \infty} T^n \vec{v}_i$$

$$= \lim_{n \to \infty} \lambda_i^n \vec{v}_i$$

$$= \left( \lim_{n \to \infty} \lambda_i^n \right) \vec{v}_i$$

However, since $|\lambda_i| > 1$, $\lim_{n \to \infty} |\lambda_i^n| \to \infty$. Thus, the system does not converge to a steady state, and $\vec{v}_i$ (or any scalar multiple of it) is an initial state such that the system does not converge.

**Common Mistakes**

• Proofs which use diagonalization must prove that $T$ is diagonalizable. In this case it is, since there are $N$ distinct eigenvalues with $N$ respective associated eigenvectors, the $N$ eigenvectors are all linearly independent, implying the matrix is diagonalizable.

• Note that the problem states that $|\lambda| > 1$. Missing the case in which $\lambda < -1$ and only addressing the scenario in which $\lambda > 1$ proves only that $\lambda > 1$ is a case for non-convergence and not the more general case of $|\lambda| > 1$.

• Many students who used diagonalization asserted that if the first component of $\vec{x}[0]$ is nonzero, then the system would not converge. The correct condition is if $V^{-1}\vec{x}[0]$ has a nonzero first component.
• When choosing $\bar{x}[0] = \sum_j \alpha_j \vec{v}_j$, it needs to be stated that $\alpha_1 \neq 0$